Spectral Analysis of Transparent Potentials

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vorgelegt von

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Declaration

I hereby declare that I wrote the thesis alone and did not receive any improper assistance. Furthermore, I have not used any materials other than those listed in the bibliography. All passages taken (in whole or in part) from these sources have been marked as citations. I have not submitted this thesis to any other institution.

Erlangen, Benedikt Wenzel

Contents

List of Symbols

$\mathbb{N}, \mathbb{R}, \mathbb{C}$	natural, real, complex numbers
$\mathscr{H}$	complex separable Hilbert space
$B(\mathscr{H})$	bounded operators on $\mathscr{H}$
$\bar{T}$	closure of a closable linear operator T
$\mathcal{B}(\mathbb{R})$	Borel σ-algebra over $\mathbb{R}$.
$\mathscr{S}(\mathbb{R}^d)$	Schwartz space (over $\mathbb{R}^d$ ($d \in \mathbb{N}$))
$C_c^\infty(\mathbb{R}^d)$	compactly supported infinitely differentiable functions (over $\mathbb{R}^d$)
$L^p(\Omega, \mu)$	standard L^p-space ($1 \leq p \leq \infty$) with measure μ over open and μ-measurable set $\Omega \subseteq \mathbb{R}^d$
$\lVert\cdot\rVert_{p,\Omega}$	standard norm in $L^p(\Omega, \mu)$
$L^p(\Omega)$	$L^p(\Omega) := L^p(\Omega, \lambda)$ with λ being the Lebesgue measure
$H^{m,p}(\Omega)$	standard Sobolev spaces with $m \in \mathbb{N}$, $1 \leq p \leq \infty$ over open and Lebesgue-measurable set $\Omega \subseteq \mathbb{R}^d$,
$\mathcal{F}$	Fourier transform (convention: $\mathcal{F}(f)(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ikx} dx$ for $f \in L^2(\mathbb{R})$)

Preamble

The aim of this thesis is to apply methods of spectral theory and functional analysis to better understand the interaction between a quantum mechanical particle and a particular family of potentials, the Pöschl-Teller potentials. The exact nature of this interaction is described by the Schrödinger equation. However, rather than describing the dynamics in as much detail as possible, one is often more interested in the following questions, which deal with the asymptotic behavior for very large times: Is it possible to somehow relate the evolution of a quantum mechanical state interacting with a potential to that of a state not interacting at all? Given an input state, what does the state look like far in the future, long after the interaction? In general, these questions are part of what is called scattering theory. The first question leads to the concept of Møller operators, the second to that of the scattering operator. Over the last few decades many interesting results have been proved in a far more general setting than what we are doing here. Many of the statements in this thesis can be found in standard textbooks on scattering theory such as [7] and [12]. Our aim is not to go into too much detail about this general theory. Instead, we will restrict ourselves to the Pöschl-Teller potential, allowing us to take a more direct approach to give an explicit expression for these two operators.

The thesis is structured as follows: The first chapter introduces some mathematical background of quantum mechanics. The second chapter is devoted to a detailed analysis of the Pöschl-Teller potentials. The third chapter builds on these results and tries to answer the two questions above.

Knowledge about topics covered in introductory lectures on functional analysis and spectral theory are prerequisites. These prerequisites can be found for example in [1],[3] or [6]. In particular the reader should be familiar with bounded and unbounded operators on Hilbert spaces and a bit theory of Sobolev spaces. Theorems proven in these lectures and used in this thesis can be found in the Appendix.

Chapter 1

A Brief Introduction to Some Mathematical Foundations of Quantum Mechanics

The first section of this chapter introduces the most fundamental operators in quantum mechanics, namely the free Hamiltonian, the momentum operator and the position operator. Furthermore, we are going to prove some properties of these operators that we will need later on. The second section deals with the analysis of bounded potentials and defines the Pöschl-Teller potential which will be studied in detail in the following chapters.

1.1 The Basic Operators of Quantum Mechanics

We start by introducing the free Hamiltonian and the momentum operator.

Definition 1.1.1. Let $D(P) = D(H_0) \coloneqq \mathscr{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$. The operators

$$P : D(P) \to L^2(\mathbb{R}), \quad f \mapsto -if'$$
$$H_0 : D(H_0) \to L^2(\mathbb{R}), \quad f \mapsto P^2 f = -f''$$

are called *free Hamiltonian* H_0 and *momentum operator* P.

If $f \in \mathscr{S}(\mathbb{R})$ then $Pf \in \mathscr{S}(\mathbb{R})$. Therefore, $H_0 = P^2$ is indeed well-defined on $D(H_0) = \mathscr{S}(\mathbb{R})$.

Remark 1.1.1. Let $\mathrm{id} : \mathbb{R} \to \mathbb{R}$ be given by $\mathrm{id}(k) = k$ and let M_{id} be the corresponding multiplication operator

$$M_{\mathrm{id}} : L^2(\mathbb{R}) \supseteq D(M_{\mathrm{id}}) \to L^2(\mathbb{R}), \qquad f \mapsto \mathrm{id} \cdot f$$

defined on

$$D(M_{\mathrm{id}}) \coloneqq \left\{ f \in L^2(\mathbb{R}) | \mathrm{id} \cdot f \in L^2(\mathbb{R}) \right\}.$$

According to the properties of the Fourier transform $\mathcal{F}$ the following relation holds

$$M_{\mathrm{id}}\big|_{\mathscr{S}(\mathbb{R})} = \mathcal{F} P \mathcal{F}^{-1}.$$

Likewise, for $\mathrm{id}^2(k) := \mathrm{id}(k)^2 = k^2$ and the multiplication operator M_{id^2} defined on $D(M_{\mathrm{id}^2}) := \left\{ f \in L^2(\mathbb{R}) | \mathrm{id}^2 \cdot f \in L^2(\mathbb{R}) \right\}$, we find

$$M_{\mathrm{id}^2}\big|_{\mathscr{S}(\mathbb{R})} = \mathcal{F} H_0 \mathcal{F}^{-1}.$$

This interplay between P, H_0, M_{id}, M_{id^2} and the Fourier transform will come in handy later on.

Regarding the essentially self-adjointness, one can check that H_0 and P are indeed essentially self-adjoint on $\mathscr{S}(\mathbb{R})$:

Lemma 1.1.2. *H_0 and P are essentially self-adjoint and their closures $\overline{H_0}$ and $\overline{P}$ satisfy $\overline{H_0} = \overline{P}^2$.*

Proof. We are going to use Lemma A.1.2 to prove the essential self-adjointness of P. Obviously, P is densely defined. Hence, we need to check that P is symmetric and that $\mathrm{Ran}(P \pm i)$ is dense. The symmetry of P follows from partial integration: Let $R > 0$ and $f \in \mathscr{S}(\mathbb{R})$. We have

$$\begin{aligned} \int_{-R}^{R} \overline{f(x)} (Pf)(x)\, dx &= -i \int_{-R}^{R} \overline{f(x)} f'(x)\, dx \\ &= \int_{-R}^{R} \overline{-if'(x)} f(x)\, dx - i \left[\overline{f(x)} f(x) \right]_{-R}^{R} \\ &= C(R) + \int_{-R}^{R} \overline{(Pf)(x)} f(x)\, dx, \end{aligned}$$

where $C(R) := -i \left[\overline{f(x)} f(x) \right]_{-R}^{R}$. Since $f \in \mathscr{S}(\mathbb{R})$, we have $\lim_{R\to\infty} C(R) = 0$. Because of $\chi_{(-R,R)} \overline{f} P f \to \overline{f} P f$ pointwise as $R \to \infty$ and $\overline{f} P f$ majorates $\chi_{(-R,R)} \overline{f} P f$, the Lebesgue dominated convergence theorem leads to

$$\langle f, Pf \rangle \xleftarrow[R\to\infty]{} \int_{-R}^{R} \overline{f(x)} (Pf)(x)\, dx = C(R) + \int_{-R}^{R} \overline{(Pf)(x)} f(x)\, dx \xrightarrow[R\to\infty]{} \langle Pf, f \rangle.$$

Now let $\phi \in \mathrm{Ran}(P \pm i)^{\perp}$ and choose $(f_j)_{j\in\mathbb{N}} \subseteq \mathscr{S}(\mathbb{R})$ such that $\mathcal{F} f_j \to \mathcal{F}\phi$ in norm. Set $g_j := \mathcal{F}^{-1} \left((\mathrm{id} \pm i)^{-1} \mathcal{F} f_j \right) \in \mathscr{S}(\mathbb{R})$. Then

$$0 = \langle \phi, (P \pm i) g_j \rangle = \langle \mathcal{F}\phi, \mathcal{F}(P \pm i) g_j \rangle = \langle \mathcal{F}\phi, \mathcal{F} f_j \rangle \xrightarrow[j\to\infty]{} \| \mathcal{F}\phi \|_{2,\mathbb{R}}^2 = \| \phi \|_{2,\mathbb{R}}^2$$

and therefore $\phi = 0$. Hence, $\overline{\mathrm{Ran}(P \pm i)} = \{0\}^{\perp} = L^2(\mathbb{R})$.

Recall that $H_0 = P^2$ by definition. Therefore,

$$\ker(H_0 \pm i) = \ker\left((P \pm i)(P \mp i) \right) \subseteq \ker(P \mp i) = \{0\}$$

which proves the essential self-adjointness of H_0. Since $\overline{P}^2$ is self-adjoint and extends H_0, we find $\overline{H_0} = \overline{P}^2$. □

As mentioned in the introduction, the goal of this thesis is to understand the scattering behavior of one specific potential, namely the Pöschl-Teller potential (a precise definition will be given in the next section). One cornerstone is going to be the spectral measures of the free Hamiltonian perturbed by that potential. In its simplest case, the Pöschl-Teller potential is identically 0. Hence, the perturbed Hamiltonian is just the free Hamiltonian. Consequently, knowing the spectral measure of the free Hamiltonian is the first step towards achieving the above-mentioned goal. The following results characterize the spectral measures of $\overline{H_0}$ and $\overline{P}$ and draw some immediate consequences.

Lemma 1.1.3. *For $S \in \mathcal{B}(\mathbb{R})$ let $E^{\overline{P}} : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R}))$ be given by*

$$E_S^{\overline{P}} \coloneqq E^{\overline{P}}(S) \coloneqq \mathcal{F}^{-1} M_{\chi_S} \mathcal{F} \quad \text{for } S \in \mathcal{B}(\mathbb{R}),$$

where χ_S denotes the characteristic function of the set S. Then $E^{\overline{P}}$ is the spectral measure of $\overline{P}$.

Proof. That $E_S^{\overline{P}}$ is an orthogonal projection follows from the unitarity of $\mathcal{F}$:

$$\left(E_S^{\overline{P}}\right)^* = (\mathcal{F})^* M_{\chi_S}^* \left(\mathcal{F}^{-1}\right)^* = \mathcal{F}^{-1} M_{\chi_S} \mathcal{F} = E_S^{\overline{P}}$$
$$\left(E_S^{\overline{P}}\right)^2 = \mathcal{F}^{-1} M_{\chi_S}^2 \mathcal{F} = \mathcal{F}^{-1} M_{\chi_S} \mathcal{F} = E_S^{\overline{P}}$$

Since $\chi_\emptyset = 0$ and $\chi_\mathbb{R} = 1$, we have $E_\emptyset^{\overline{P}} = 0$ and $E_\mathbb{R}^{\overline{P}} = 1$ respectively. If $(S_j)_{j\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R})$ is a sequence of Borel measurable sets satisfying $S_i \cap S_j = \emptyset$ for $i \neq j$, then $\sum_{j\in\mathbb{N}} \chi_{S_j} = \chi_{\bigcup_{j\in\mathbb{N}} S_j}$ and $\sum_{j\in\mathbb{N}} E_{S_j}^{\overline{P}} f = E_{\bigcup_{j\in\mathbb{N}} S_j}^{\overline{P}} f$ for all $f \in L^2(\mathbb{R})$. Therefore, $E^{\overline{P}}$ is a projection valued measure. The measures $\mu_f(S) \coloneqq \langle f, E_S^{\overline{P}} f\rangle$ for $f \in L^2(\mathbb{R})$ evaluate to

$$\mu_f^{\overline{P}}(S) = \langle \mathcal{F} f, \chi_S \mathcal{F} f\rangle = \int_S |\mathcal{F} f|^2 \, d\lambda,$$

so that $\mu_f^{\overline{P}}$ is absolutely continuous with respect to the Lebesgue measure λ and $\frac{d\mu_f^{\overline{P}}}{d\lambda} = |\mathcal{F} f|^2$. Define the self-adjoint operator

$$T \coloneqq \int_\mathbb{R} \lambda \, dE^{\overline{P}}(\lambda).$$

We show that T extends P. To this end, choose $\phi \in \mathscr{S}(\mathbb{R})$ arbitrarily. One finds

$$\begin{aligned}
\langle f, Pf\rangle &= \langle \mathcal{F} f, \mathcal{F}(Pf)\rangle \\
&= \int_\mathbb{R} x |(\mathcal{F} f)(x)|^2 \, d\lambda(x) \\
&= \int_\mathbb{R} x \, d\mu_f^{\overline{P}}(x) \\
&= \langle f, Tf\rangle.
\end{aligned}$$

By the polarization identity this generalizes to $\langle f, Pg\rangle = \langle f, Tg\rangle$ for all $f, g \in \mathscr{S}(\mathbb{R})$ and since $\mathscr{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is dense, $Pg = Tg$ for all $g \in \mathscr{S}(\mathbb{R})$. Therefore, the self-adjoint operator T extends P indeed. Since P is essentially self-adjoint (Lemma 1.1.2), this implies $\overline{P} = T$. The claim follows from the uniqueness of the spectral measure. □

Recall that $\overline{H_0} = \overline{P}^2$ (Lemma 1.1.2). Using functional calculus we easily obtain the spectral measures of $\overline{H_0}$:

Theorem 1.1.4. *For $S \in \mathcal{B}(\mathbb{R})$ define $\xi_S(x) = \chi_S(x^2)$ and let $E^0 : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R}))$ be given by*

$$E^0_S \coloneqq E^0(S) \coloneqq \mathcal{F}^{-1} M_{\xi_S} \mathcal{F} \quad \textit{for } S \in \mathcal{B}(\mathbb{R}).$$

Then E^0 is the spectral measure of $\overline{H_0}$.

Proof. For $S \in \mathcal{B}(\mathbb{R})$ we have $\chi_S(\overline{H_0}) = \chi_S(\overline{P}^2) = (\chi_S \circ f)(\overline{P})$ where $f(p) = p^2$. Since $\chi_S \circ f = \chi_{f^{-1}(S)}$, we find $\chi_S(\overline{H_0}) = E^{\overline{P}}_{f^{-1}(S)}$ and Lemma 1.1.3 yields

$$\chi_S(\overline{H_0}) = \mathcal{F}^{-1} M_{\chi_{f^{-1}(S)}} \mathcal{F} = \mathcal{F}^{-1} M_{\xi_S} \mathcal{F} = E^0_S.$$

□

With a bit of theory of Sobolev spaces and weak derivatives we can conclude what the domains of $\overline{P}$ and $\overline{H_0}$ are and how these extensions act on weak differentiable functions.

Corollary 1.1.5. *The closure $\overline{H_0}$ of H_0 is defined on $D(\overline{H_0}) = H^{2,2}(\mathbb{R})$ and given by $\overline{H_0} f = -f''$ for $f \in H^{2,2}(\mathbb{R})$. Likewise, the closure $\overline{P}$ of P is defined on $D(\overline{P}) = H^{1,2}(\mathbb{R})$ and given by $\overline{P} f = -if'$ for $f \in H^{1,2}(\mathbb{R})$.*

Proof. By definition, $f \in D(\overline{P})$ if and only if $\int_{\mathbb{R}} \lambda^2 \, d\mu^{\overline{P}}_f = \int_{\mathbb{R}} x^2 |\mathcal{F} f(x)|^2 < \infty$ and hence $f \in H^{1,2}(\mathbb{R})$. Furthermore, Lemma 1.1.3 implies $\overline{P} f = \mathcal{F}^{-1}(\mathrm{id}_{\mathbb{R}} \mathcal{F}(f)) = -i\mathcal{F}^{-1}(i \mathrm{id}_{\mathbb{R}} \mathcal{F}(f)) = -if'$ for $f \in D(\overline{P})$, where the last equality is just the definition of weak derivatives. Since $\overline{H_0} = \overline{P}^2$, the domain $D(\overline{H_0})$ consists of all $f \in H^{1,2}(\mathbb{R}) = D(\overline{P})$ for which $f' \in H^{1,2}(\mathbb{R}) = D(\overline{P})$. Therefore, $D(\overline{P}) = H^{2,2}(\mathbb{R})$ and $\overline{H_0} f = (-i)^2 f'' = -f''$ for $f \in H^{2,2}(\mathbb{R})$. □

After investigation of self-adjointness, the next natural question arising in physics addresses the spectrum of the Hamiltonian. Clearly, we have to understand the spectrum of the free Hamiltonian first, before dealing with perturbed ones. As we already know the spectral measures, we could just use that for any self-adjoint operator T the relation

$$\mathrm{supp}(E^T) = \sigma(T)$$

holds, to conclude that

$$\sigma(\overline{P}) = \mathbb{R} \quad \text{and} \quad \sigma(\overline{H_0}) = [0, \infty).$$

However, in the case of $\overline{H_0}$, an alternative way using approximate eigenvalues is presented, as we will reuse this result later on and build up on that idea.

Lemma 1.1.6. *The spectrum of $\overline{H_0}$ is given by $\sigma(\overline{H_0}) = \mathbb{R}_{\geq 0}$.*

Proof. Let $\lambda > 0$. We are going to prove that λ is an approximate eigenvalue and use Lemma A.1.3 to conclude $\lambda \in \sigma(\overline{H_0})$. In that regard, we notice that $f(x) = e^{i\sqrt{\lambda}x}$ solves $(-\frac{d^2}{dx^2} - \lambda) f = 0$. For $n \in \mathbb{N}$ choose $\eta_n \in C^\infty_c(\mathbb{R})$ such that

$$0 \leq \eta_n \leq 1, \quad \eta_n\big|_{(-n,n)} = 1, \quad \mathrm{supp}\, \eta_n \subseteq [-n-1, n+1]$$

and such that there are constants $C_1, C_2 > 0$ with

$$|\eta_n'| \leq C_1 \quad \text{and} \quad |\eta_n''| \leq C_2 \forall n \in \mathbb{N}.$$

Now define

$$g_n := \frac{\eta_n f}{\|\eta_n f\|_{2,\mathbb{R}}} \in D(\overline{H_0})$$

and verify that these functions form a Weyl-sequence for λ ($\Omega := \operatorname{supp}(\eta_n)\backslash(-n,n)$):

$$\begin{aligned}
&\|(\overline{H_0}-\lambda)g_n\|_{2,\mathbb{R}}^2 \\
&= \|\eta_n f\|_{2,\mathbb{R}}^{-2} \int_{\mathbb{R}} |(\overline{H_0}-\lambda)\eta_n f|^2\,dx \\
&= \|\eta_n f\|_{2,\mathbb{R}}^{-2} \left(\int_{(-n,n)} |(\overline{H_0}-\lambda)f|^2\,dx + \int_{\Omega} |(\overline{H_0}-\lambda)\eta_n f|^2\,dx\right) \\
&= \|\eta_n f\|_{2,\mathbb{R}}^{-2} \int_{\Omega} |-(\eta_n'' f + 2\eta_n' f' + \eta_n f'') - \lambda\eta_n f|^2\,dx \\
&= \|\eta_n f\|_{2,\mathbb{R}}^{-2} \int_{\Omega} |-\eta_n'' - 2i\sqrt{\lambda}\eta_n'|^2 |f|^2\,dx \\
&\le \|\eta_n f\|_{2,\mathbb{R}}^{-2} |\Omega| (C_2^2 + 4\lambda C_1^2) \le \|\eta_n f\|_{2,\mathbb{R}}^{-2}(2C_2^2 + 8\lambda C_1^2)
\end{aligned}$$

and since $\|\eta_n f\|_{2,\mathbb{R}} \ge \sqrt{2n}$, we have $\|(\overline{H_0}-\lambda)g_n\|_{2,\mathbb{R}} \to 0$ for $n \to \infty$. Since $\|g_n\|_{2,\mathbb{R}} = 1$ this calculation shows that λ is an approximate eigenvalue. As $\sigma(\overline{H_0})$ is closed, the inclusion $\mathbb{R}_{\ge 0} \subseteq \sigma(\overline{H_0})$ follows. It remains to prove $(-\infty,0)\cap\sigma(\overline{H_0}) = \emptyset$. However, this is clear since $\langle f, \overline{H_0} f\rangle = \langle Pf, Pf\rangle \|f'\|_{2,\mathbb{R}}^2 \ge 0$ ($f \in D(\overline{H_0})$). □

Another consequence of Theorem 1.1.4 and Lemma 1.1.3 is the following:

Corollary 1.1.7. *The spectra $\sigma(\overline{H_0})$ and $\sigma(\overline{P})$ are both purely absolutely continuous. In particular $\overline{H_0}$ and $\overline{P}$ do not have eigenvalues.*

Proof. In the case of $\overline{P}$, this follows from the proof of Lemma 1.1.3, where we explicitly constructed the measures $\mu_f^{\overline{P}}$. In the case of $\overline{H_0}$, the measures $\mu_f^0(S) := \langle f, E_S^0 f\rangle$ are given by

$$\begin{aligned}
\mu_f^0(S) = \langle \mathcal{F}f, \xi_S \mathcal{F}f\rangle &= \int_{\mathbb{R}} \chi_S(x^2)|\mathcal{F}f(x)|^2\,d\lambda(x) \\
&= \int_{\mathbb{R}_+} \chi_S(x)\frac{1}{2\sqrt{x}}\left(|\mathcal{F}f(\sqrt{x})|^2 + |\mathcal{F}f(-\sqrt{x})|^2\right)\,d\lambda(x).
\end{aligned}$$

Hence, μ_f^0 is absolutely continuous with respect to λ and

$$\frac{d\mu_f^0}{d\lambda}(x) = \chi_{\mathbb{R}_+}(x)\frac{|\mathcal{F}f(\sqrt{x})|^2 + |\mathcal{F}f(-\sqrt{x})|^2}{2\sqrt{x}}.$$

□

We finish this introductory section with a few words on unitary equivalence and Remark 1.1.1. Recall that two operators $T : D(T) \to H$ and $S : D(S) \to H$ are said to be unitarily equivalent if there exists a unitary operator $U \in B(H)$ such that $D(T) = UD(S)$ and $T = USU^*$. Unitary equivalence preserves many properties as the following theorem shows.

Theorem 1.1.8. *Let $U \in B(\mathscr{H})$ be unitary and $T : \mathscr{H} \supseteq D(T) \to \mathscr{H}$ be an essentially self-adjoint operator. Then $S := UTU^* : \mathscr{H} \supseteq UD(T) \to \mathscr{H}$ is also essentially self-adjoint. Furthermore, if T is self-adjoint, then S is self-adjoint and for every Borel measurable function $f : \mathbb{R} \to \mathbb{C}$ one has $f(S) = Uf(T)U^*$.*

Proof. First, suppose that T is essentially self-adjoint. Because of the unitarity of U, the domain $D(S) := UD(T) \subseteq \mathscr{H}$ is dense. For all $x, y \in UD(T)$ we have

$$\langle x, Sy\rangle = \langle U^*x, TU^*y\rangle = \langle TU^*x, U^*y\rangle = \langle UTU^*x, y\rangle = \langle Sx, y\rangle,$$

where we used $U^*x, U^*y \in D(T)$ and the symmetry of T in the second equality. Thus, S is symmetric. Pick $y \in \mathscr{H}$. There exists a sequence $(x_j^\pm)_{j\in\mathbb{N}} \subset D(T)$ such that $(T \pm i)x_j^\pm \to U^*y$. Consider the sequence $\tilde{x}_j^\pm := Ux_j$. One has $\tilde{x}_j^\pm \in D(S)$ and

$$(S \pm i)\tilde{x}_j^\pm = U(T \pm i)U^*Ux_j^\pm = U(T \pm i)x_j^\pm \xrightarrow[j\to\infty]{} UU^*y = y.$$

Since $y \in \mathscr{H}$ was arbitrary, S is essentially self-adjoint (see Lemma A.1.2). Now suppose that T is self-adjoint. Following the same argument, one shows that $\mathrm{Ran}(S \pm i) = \mathscr{H}$. Again, consider $y \in \mathscr{H}$ and choose $x^\pm \in D(T)$ such that $(T \pm i)x^\pm = Uy$. Then $\tilde{x}^\pm := Ux^\pm \in D(S)$ and $(S \pm i)\tilde{x}^\pm = y$.

Next suppose that T is self-adjoint and therefore, by the above arguments, S is self-adjoint too. Denote with E^T, E^S their spectral measures. Because of the unitarity of U, $E_V := UE_V^TU^*$ (for $V \in \mathcal{B}(\mathbb{R})$) is a projection valued measure. Define the operator $\tilde{S} := \int_{\mathbb{R}} \lambda\, dE(\lambda)$. Since $\mu_{Uy}(V) := \langle Uy, E_V Uy\rangle = \langle y, E_V^T y\rangle =: \mu_y^T(V)$ (for every $y \in \mathscr{H}$) one has $y \in D(T)$ if and only if $Uy \in D(\tilde{S})$. Therefore, $UD(T) = D(\tilde{S})$ and for $x \in D(T)$ we have

$$\langle x, Tx\rangle = \int_{\mathbb{R}} \lambda\, d\mu_x^T(\lambda) = \int_{\mathbb{R}} \lambda\, d\mu_{Ux}(\lambda) = \langle Ux, \tilde{S}Ux\rangle.$$

Thanks to polarization this implies $\langle y, Tx\rangle = \langle y, U^*\tilde{S}Ux\rangle$ for all $x, y \in D(T)$. Since $D(T)$ is dense, $Tx = U^*\tilde{S}Ux$ for all $x \in D(T)$. Therefore, $S = \tilde{S}$ and by uniqueness of the spectral measure $E = E^S$. Therefore, $\chi_V(S) = U\chi_V(T)U^*$ for measurable $V \in \mathcal{B}(\mathbb{R})$ and consequently $\phi(S) = U\phi(T)U^*$ for step functions ϕ. Hence, $f(S) = Uf(T)U^*$ for all Borel measurable f. □

In the subsequent chapters we will need a slightly reformulated version:

Scholium 1.1.9. *Let $U \in B(\mathscr{H})$ be unitary, $T : \mathscr{H} \supseteq D(T) \to \mathscr{H}$ a self-adjoint operator and E^T the associated projection valued measure. Then $E_S := UE_S^TU^*$ defines a projection valued measure and one has*

$$UTU^* = \int_{\mathbb{R}} \lambda\, dE_\lambda.$$

Remark 1.1.10. Recall that, according to Remark 1.1.1, P and M_{id} are related via the Fourier transform. According to the previous theorem and the essential self-adjointness of P (Lemma 1.1.2), the operator $X := M_{\mathrm{id}}\big|_{\mathscr{S}(\mathbb{R})}$ is essentially self-adjoint with self-adjoint closure $\overline{X} = \mathcal{F}^{-1}\overline{P}\mathcal{F}$. X is called position operator. For sure, we could have done everything in reverse and first introduce the position operator, prove self-adjointness, determine the spectral measure of $\overline{X}$ and then introduce the momentum operator and the free Hamiltonian and deduce all their above-mentioned properties from the properties of X. However, this way was not chosen as it would have given not that many insights into the behavior of H_0 and P.

1.2 Bounded Potentials

So far we investigated the spectrum and spectral measures of the free Hamiltonain. The next step towards our goal is to understand the effect of bounded perturbations of the free Hamiltonian. Recall that for bounded $V \in B(\mathscr{H})$ and not necessarily bounded $T : \mathscr{H} \supseteq D(T) \to \mathscr{H}$ the sum $T+V$ is defined as the operator

$$T+V : D(T+V) \to \mathscr{H}, \quad (T+V)x := Tx + Vx \quad \forall x \in D(T+V),$$

where $D(T+V) := D(T)$. By a perturbed Hamiltonian we mean the operator H_0+V for a bounded $V \in B(\mathscr{H})$, called the perturbation. If V is a multiplication operator by some function $v \in L^\infty(\mathbb{R})$ we call v (or V) a potential. These perturbations might change the spectrum (e.g. by introducing point spectrum), but at least we can hope that (essentially) self-adjoint perturbations do not change (essential) self-adjointness. The following result formalizes our hope:

Theorem 1.2.1. *Let $T : \mathscr{H} \supseteq D(T) \to \mathscr{H}$ be an essentially self-adjoint operator and $V = V* \in B(\mathscr{H})$. Then $T+V : D(T) \to \mathscr{H}$ is essentially self-adjoint. Furthermore, if T is self-adjoint, then $T+V$ is also self-adjoint.*

Before proving this theorem we need a small supplementary lemma.

Lemma 1.2.2. *Let $T : \mathscr{H} \supseteq D(T) \to \mathscr{H}$ be a densely defined and closable operator and $V \in B(\mathscr{H})$. Then $T+V : D(T) \to \mathscr{H}$ is closable with closure $\overline{T}+V$ and $(T+V)^* = T^* + V^*$.*

Proof. To see that $\overline{T}+V$ is closed, consider $(x_j)_{j\in\mathbb{N}} \subseteq D(\overline{T}+V) = D(\overline{T})$ such that there are $x, y \in \mathscr{H}$ with $x_j \to x$ and $(\overline{T}+V)x_j \to y$. Then, since V is continuous, $x_j \to x$ and $\overline{T}x_j \to y - Vx$. Therefore, since $\overline{T}$ is closed, $x \in D(\overline{T})$ and $\overline{T}x = y - Vx$. Consequently, we have $x \in D(\overline{T}+V)$ as well as $(\overline{T}+V)x = y$, which shows that $\overline{T}+V$ is closed. Clearly, $\overline{T}+V$ is a closed extension of $T+V$. Hence, $T+V$ is closable and we have to show that $\overline{T}+V = \overline{T+V} =: B$. Consider the operator $B-V$ which is closed according to the arguments above. Furthermore, for $x \in D(T)$ we have $(B-V)x = (T+V-V)x = Tx$ so that $B-V$ is a closed extension of T and therefore $\overline{T} \subseteq B - V$. Therefore, $\overline{T}+V \subseteq B = \overline{T+V}$ which proves the claim.

Next we are going to prove $(T+V)^* = T^* + V^*$. By the continuity of V we find

$$\begin{aligned} x \in D(T^*) &\iff D(T) \ni y \mapsto \langle x, Ty \rangle \text{ is continuous} \\ &\iff D(T) \ni y \mapsto \langle x, Ty \rangle + \langle x, Vy \rangle \text{ is continuous} \\ &\iff x \in D((T+V)^*) \end{aligned}$$

so that $D(T^* + V^*) = D(T^*) = D((T+V)^*)$. Now the relation $\langle x, (T+V)y \rangle = \langle x, Ty \rangle + \langle x, Vy \rangle = \langle (T^* + v^*)x, y \rangle$ for $x \in D(T^*), y \in D(T)$ concludes the proof. □

Equipped with Lemma 1.2.2 we are ready to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Let T be essentially self-adjoint. Then $T+V$ is symmetric (since T and V are both symmetric) and $\overline{T+V}^* = (\overline{T}+V)^* = \overline{T}^* + V^* = \overline{T} + V = \overline{T+V}$, where we used Lemma 1.2.2. Hence, $\overline{T+V}$ is self-adjoint. Therefore, $T+V$ is essentially self-adjoint. If T is self-adjoint, then, again by Lemma 1.2.2, $(T+V)^* = T^* + V^* = T + V$ so that $T+V$ is self-adjoint. □

In particular, we are interested in applying this theorem to Schrödinger operators of the form $H = H_0 + V$, where V is a bounded (real valued) potential. In that specific case we are able to say something about the spectrum of the perturbed Hamiltonian H.

Lemma 1.2.3. *Let $v \in L^\infty(\mathbb{R},\mathbb{R})$ and $V \in B(L^2(\mathbb{R}))$ the corresponding multiplication operator by v. Set $H := \overline{H_0} + V : D(\overline{H_0}) \to L^2(\mathbb{R})$. Then:*

1. $(-\infty, \operatorname{ess\,inf}(v)) \cap \sigma(H) = \emptyset$
2. *If there exists $C \in \mathbb{R}$ such that $\|\chi_{(n,\infty)}(v - C)\|_{\infty,\mathbb{R}} \to 0$ as $n \to \infty$ then $(C,\infty) \subseteq \sigma(H)$. Likewise, if there exists $\tilde{C} \in \mathbb{R}$ such that $\|\chi_{(-\infty,n)}(v - \tilde{C})\|_{\infty,\mathbb{R}} \to 0$ as $n \to -\infty$ then $(\tilde{C},\infty) \subseteq \sigma(H)$.*

Proof. Notice that, since v is real-valued, V is self-adjoint. Furthermore, according to Theorem 1.2.1 and the self-adjointness of $\overline{H_0}$ (Lemma 1.1.2), H is self-adjoint.

1. Let $\lambda < \operatorname{ess\,inf}(v)$, $\phi \in D(\overline{H_0})$ and set $C := \operatorname{ess\,inf}(v - \lambda) > 0$. We observe that

$$\begin{aligned}\|(H-\lambda)\phi\|_{2,\mathbb{R}}\|\phi\|_{2,\mathbb{R}} &\geq \langle (H-\lambda)\phi, \phi\rangle \\ &= \langle (V-\lambda)\phi,\phi\rangle + \langle \overline{H_0}\phi,\phi\rangle \\ &\geq \operatorname{ess\,inf}(v-\lambda)\|\phi\|_{2,\mathbb{R}}^2 = C\|\phi\|_{2,\mathbb{R}}^2.\end{aligned}$$

   Thus, $C\|\phi\|_{2,\mathbb{R}} \leq \|(H-\lambda)\phi\|_{2,\mathbb{R}}$ for all $\phi \in D(\overline{H_0})$. Therefore, $H-\lambda$ is injective and $\operatorname{Ran}(H-\lambda)$ is closed. Moreover, $\ker(H-\lambda) = \{0\}$ also implies that

$$\begin{aligned}\{0\} = \ker(H-\lambda) &= \ker((H-\lambda)^*) = \operatorname{Ran}(H-\lambda)^\perp \\ \implies L^2(\mathbb{R}) &= (\operatorname{Ran}(H-\lambda)^\perp)^\perp = \overline{\operatorname{Ran}(H-\lambda)}.\end{aligned}$$

   But because $\operatorname{Ran}(H-\lambda)$ is closed, we find $\operatorname{Ran}(H-\lambda) = L^2(\mathbb{R})$ and thus $\lambda \notin \sigma(H)$.

2. Let $\epsilon > 0$, set $\lambda := C + \epsilon > C$ and assume that $\|\chi_{(n,\infty)}(v - C)\|_{\infty,\mathbb{R}} \to 0$ as $n \to \infty$. We are going to show that λ is an approximate eigenvalue of H (see Lemma A.1.3). To do so, choose a sequence $(\phi_j)_{j\in\mathbb{N}} \subseteq D(\overline{H_0})$ such that

$$\|\phi_j\|_{2,\mathbb{R}} = 1, \quad \|(\overline{H_0} - \epsilon)\phi_j\|_{2,\mathbb{R}} \to 0, \quad \operatorname{supp}(\phi_j) \subseteq [-j-1, j+1] = B_{j+1}(0)$$

   (such a sequence has been given explicitly in the proof of Lemma 1.1.6). Thanks to the asymptotic behavior of v at infinity we can find a sequence of points $(p_j)_{j\in\mathbb{N}} \in \mathbb{R}$ such that

$$\|\chi_{B_{j+1}(p_j)}(v - C)\|_{\infty,\mathbb{R}} \leq \frac{1}{j} \quad \forall j \in \mathbb{N}.$$

   Now we can shift each ϕ_j by p_j and obtain a sequence $(g_j)_{j\in\mathbb{N}} \subseteq D(\overline{H_0})$ given by $g_j := \phi_j \circ \tau^{p_j}$ where $\tau^{p_j}(x) = x - p_j$ implements the shift by p_j. The sequence $(g_j)_{j\in\mathbb{N}}$ has the following properties:

   (a) $\|g_j\|_{2,\mathbb{R}} = \|\phi_j\|_{2,\mathbb{R}} = 1$ and $\operatorname{supp}(g_j) \subseteq B_{j+1}(p_j)$.

(b) Using $g_j'' = \phi_j'' \circ \tau^{p_j}$ we get $\overline{H_0} g_j = (\overline{H_0}\phi_j) \circ \tau^{p_j}$ and therefore

$$\|(\overline{H_0} - \epsilon)g_j\|_{2,\mathbb{R}} = \|(\overline{H_0} - \epsilon)\phi_j\|_{2,\mathbb{R}} \to 0.$$

(c) By definition of p_j we have

$$\begin{aligned}\|(V - C)g_j\|_{2,\mathbb{R}} &= \|\chi_{B_{j+1}(p_j)}(v - C)g_j\|_{2,\mathbb{R}} \\ &\leq \|\chi_{B_{j+1}(p_j)}(v - C)\|_{\infty,\mathbb{R}}\|g_j\|_{2,\mathbb{R}} \\ &\leq \frac{1}{j} \xrightarrow[j\to\infty]{} 0.\end{aligned}$$

These three properties together imply that λ is indeed an approximate eigenvalue of H:

$$\|(H - \lambda)g_j\|_{2,\mathbb{R}} \leq \|(\overline{H_0} - \epsilon)g_j\|_{2,\mathbb{R}} + \|(V - C)g_j\|_{2,\mathbb{R}} \xrightarrow[j\to\infty]{} 0$$

The proof in the case of $\|\chi_{(-\infty,n)}(v - \tilde{C})\|_{\infty,\mathbb{R}} \to 0$ as $n \to -\infty$ works exactly the same.

□

Remark 1.2.4. The fact that there are no parts of the spectrum below the potentials minimum is, from a physical point of view, not very surprising, as elements in $\sigma(H)$ correspond to possible energies. The total energy consists of a kinetic part and potential energy, where the latter is given by the potential v. As the kinetic energy is always positive, the sum of kinetic and potential energy can never be smaller than the infimum of v. The second part of the Lemma shows that the spectrum heavily depends on the asymptotic behavior of the potential at $\pm\infty$.

The question that remains is this: What parts (if any) of the spectrum lie in between $\operatorname{ess\,inf}(v)$ and $C, \tilde{C}$ and what type of spectrum do we have (e.g. pure point, absolute continuous or even singular continuous)? In general, these questions can be quite difficult to answer, as there are potentials with rather "bizarre" spectra (more on this can be found, for example, in [8]). However, in the case of the Pöschl-Teller potential, we will see in the next chapter that the spectrum is "well behaved" and we do not have to deal with anything strange.

Chapter 2

The Pöschl-Teller Potential

In this chapter we finally introduce the family H_l of Hamiltonians, which we want to study in more detail. We are going to derive their full spectral decomposition. At first, we are going to determine the point spectrum and in particular we are interested in the relation between $\sigma_p(H_{l-1})$ and $\sigma_p(H_l)$. This knowledge will be helpful when looking at the spectral measures of H_l in a next step.

2.1 Analysis of the Spectrum

Definition 2.1.1. For $l \in \mathbb{N} \cup \{0\}$ the map

$$v_l : \mathbb{R} \to \mathbb{R}, \quad x \mapsto -\frac{l(l+1)}{\cosh^2(x)}$$

is called the *Pöschl-Teller potential*. Furthermore, we define $V_l \in B(L^2(\mathbb{R}))$ to be the multiplication operator by v_l

$$V_l : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad f \mapsto f \cdot v_l$$

and we define the operator H_l respectively $\overline{H_l}$ to be

$$H_l : D(H_0) \to L^2(\mathbb{R}), \quad f \mapsto (H_0 + V_l)f$$
$$\overline{H_l} : D(\overline{H_0}) \to L^2(\mathbb{R}), \quad f \mapsto (\overline{H_0} + V_l)f.$$

Since $v_l \in L^\infty(\mathbb{R})$, V_l is indeed a bounded potential. Therefore, we can apply the work we did in the previous section to draw some immediate consequences:

1. According to Theorem 1.2.1, H_l is essentially self-adjoint and $\overline{H_l}$ is its self-adjoint closure (which justifies the notation).
2. Regarding Theorem 1.2.3, we have
   (a) $(0, \infty) \subseteq \sigma(H_l)$ and, since the spectrum is closed, $\sigma(\overline{H_0}) = [0, \infty) \subseteq \sigma(\overline{H_l})$.
   (b) $(-\infty, -l(l+1)) \subseteq \rho(H_l)$.

A few words on the physical importance of the Pöschl-Teller potential: In Figure 2.1 a plot of the Pöschl-Teller potential for $l \in \{1, 2, 3\}$ is given. The Pöschl-Teller potential has already been used in molecular physics and chemistry about 30 years ago ([14]). Recently, a generalized version extending Definition 2.1.1 was applied for better understanding the interactions and vibrational energies of molecules ([5]).

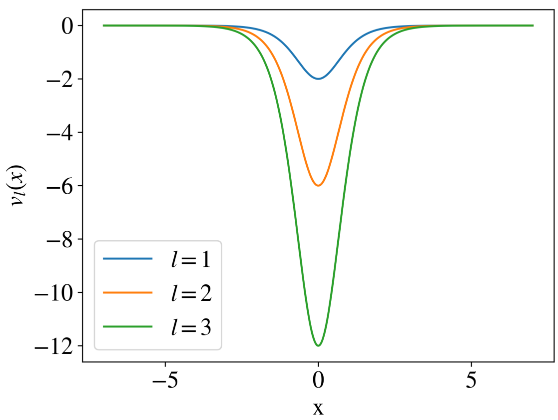

Figure 2.1: Plot of the Pöschl-Teller Potential v_l for different l.

As mentioned above, our first goal is to determine the point spectrum. We are going to need the following operators:

Definition 2.1.2. Let $l \in \mathbb{N} \cup \{0\}$ and set $D(a_l) := D(P) = \mathscr{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$. Let $M_{th} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the multiplication operator by tanh. Then the linear operator $a_l : D(a_l) \to L^2(\mathbb{R})$ is defined by

$$a_l f(x) := (Pf)(x) - il(M_{th} f)(x).$$

Lemma 1.2.2 allows us to obtain the closure of a_l and its adjoint.

Corollary 2.1.1. *For each $l \in \mathbb{N} \cup \{0\}$ the operator a_l is closable with closure $\overline{a_l} = \overline{P} - ilM_{th}$ and its adjoint is given by $a_l^* = \overline{P} + ilM_{th}$.*

Proof. Observe that tanh is bounded, real-valued and therefore $M_{th}^* = M_{th} \in B(H)$. Hence, the claim is a direct consequence of Lemma 1.2.2 and the essential self-adjointness of P (Lemma 1.1.2). □

Furthermore, we are going to need compositions of a_l and a_l^*. Therefore, we define:

Definition 2.1.3. Let $l \in \mathbb{N} \cup \{0\}$ and set $D(A_l) := D(B_l) := \mathscr{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$. We define the linear operators $A_l : D(A_l) \to L^2(\mathbb{R})$ and $B_l : D(B_l) \to L^2(\mathbb{R})$ to be

$$A_l := a_l^* a_l \quad \text{and} \quad B_l := a_l a_l^*.$$

Remark 2.1.2. If $f \in \mathscr{S}(\mathbb{R})$ then $a_l f \in \mathscr{S}(\mathbb{R}) \subseteq D(a_l^*)$ and $a_l^* f \in \mathscr{S}(\mathbb{R}) = D(a_l)$. Hence, the operators A_l and B_l are well-defined on $\mathscr{S}(\mathbb{R})$.

The following useful relations between A_l, B_l and H_l hold:

Lemma 2.1.3. *Let $l \in \mathbb{N}$. Then the following relations hold:*

1. $H_l = A_l - l^2$
2. $H_{l-1} = B_l - l^2$

Proof. For $f \in \mathscr{S}(\mathbb{R})$ we have (using Corollary 2.1.1) $a_l^* f = (\overline{P} + ilM_{th})f = (P + ilM_{th})f$. Therefore, the relations follow from a direct calculation. Indeed, for $f \in \mathscr{S}(\mathbb{R})$ we have

$$\begin{aligned} A_l f &= (P + ilM_{th})(-if' - il\tanh\cdot f) \\ &= -f'' - l(\tanh\cdot f)' + l\tanh\cdot f' + l^2\tanh^2\cdot f \\ &= H_0 f - l(\tanh\cdot f' + \frac{1}{\cosh^2}f) + l\tanh\cdot f' + l^2(1 - \frac{1}{\cosh^2})f \\ &= H_0 f - \frac{l(l+1)}{\cosh^2}f + l^2 f \\ &= H_l f + l^2 f. \end{aligned}$$

And likewise we find (again $f \in \mathscr{S}(\mathbb{R})$)

$$\begin{aligned} B_l f &= (P - ilM_{th})(-if' + il\tanh\cdot f) \\ &= -f'' + l(\tanh\cdot f)' - l\tanh\cdot f' + l^2\tanh^2\cdot f \\ &= H_0 f + l(\tanh\cdot f' + \frac{1}{\cosh^2}f) - l\tanh\cdot f' + l^2(1 - \frac{1}{\cosh^2})f \\ &= H_0 f - \frac{l(l-1)}{\cosh^2}f + l^2 f \\ &= H_{l-1} f + l^2 f. \end{aligned}$$

□

As we already know that H_l is essentially self-adjoint and hence closable, we can use Lemma 1.2.2 to conclude that the operators A_l, B_l are closable and we are able to obtain their closures.

Corollary 2.1.4. *Let $l \in \mathbb{N}$. Then A_l and B_l are closable with closures $\overline{A_l} = a_l^*\overline{a_l}$ and $\overline{B_l} = \overline{a_l}a_l^*$ defined on $D(\overline{A_l}) = D(\overline{B_l}) = H^{2,2}(\mathbb{R})$. Furthermore, we have:*

1. $\overline{H_l} = \overline{A_l} - l^2$
2. $\overline{H_{l-1}} = \overline{B_l} - l^2$

Proof. By Lemma 1.2.2 we have that A_l and B_l are closable with closures $\overline{A_l} = \overline{H_l} + l^2$ and $\overline{B_l} = \overline{H_{l-1}} + l^2$ (with domain $D(\overline{A_l}) = D(\overline{B_l}) = D(\overline{H_0}) = H^{2,2}(\mathbb{R})$). By the same calculations as in the previous proof we find that

$$\overline{a_l}^*\overline{a_l} = \overline{H_l} + l^2 \implies \overline{A_l} = \overline{a_l}^*\overline{a_l} = a_l^*\overline{a_l}$$

and

$$\overline{a_l}\,\overline{a_l}^* = \overline{H_{l-1}} + l^2 \implies \overline{B_l} = \overline{a_l}\,\overline{a_l}^* = \overline{a_l}a_l^*.$$

Notice, that in order to actually apply the calculations in the previous proof, we have to make sure that we are allowed to apply the product rule on the term $\overline{P}(\tanh f)$ for $f \in H^{2,2}(\mathbb{R})$. Clearly, $\tanh\cdot f \in L^2(\mathbb{R})$ because of $\tanh \in L^\infty(\mathbb{R})$. Let $\phi \in C_c^\infty(\mathbb{R})$. Then $\tanh\cdot\phi \in C_c^\infty(\mathbb{R})$ since $\tanh \in C^\infty(\mathbb{R})$ and we have

$$\begin{aligned} &-\int_{\mathbb{R}} f\cdot(\tanh\cdot\phi)'\,d\lambda = \int_{\mathbb{R}} f'\cdot(\tanh\cdot\phi)\,d\lambda \\ \iff &-\int_{\mathbb{R}}(f\cdot\tanh)\cdot\phi'\,d\lambda = \int_{\mathbb{R}}(f\cdot\tanh' + f'\cdot\tanh)\cdot\phi\,d\lambda. \end{aligned}$$

Since $f \cdot \tanh' + f' \cdot \tanh \in L^2(\mathbb{R})$ we can conclude $\tanh \cdot f \in H^{1,2}(\mathbb{R})$ with weak derivative $(\tanh \cdot f)' = f \cdot \tanh' + f' \cdot \tanh$. $\square$

Lemma 2.1.5. *Let* $l \in \mathbb{N} \cup \{0\}$. *Then* $\ker(a_l^*) = \{0\}$.

Proof. Recall that, according to Corollary 2.1.1, we have $a_l^* = \overline{P} + ilM_{th} : D(\overline{P}) = H^{1,2}(\mathbb{R}) \to L^2(\mathbb{R})$ so that we are looking for weak solutions $f \in H^{1,2}(\mathbb{R})$ of

$$-if' + il \tanh \cdot f = 0 \iff f' = l \tanh \cdot f. \tag{2.1}$$

Suppose that we have a weak solution $f \in H^{1,2}(\mathbb{R})$ of equation 2.1. By the argument we already did in the proof of Corollary 2.1.4, we have $\tanh f \in H^{1,2}(\mathbb{R})$. Equation 2.1 then implies $f' \in H^{1,2}(\mathbb{R})$. Consequently, we have $f \in H^{2,2}(\mathbb{R})$. Thanks to the Sobolev embedding theorem (Appendix A.2.1) we can conclude that f is a classical solution. Each classical solution of equation 2.1 is of the form

$$f(x) = f(0)e^{l \int_0^x \tanh\, dx}$$

for some $f(0) \in \mathbb{C}$. If $f(0) \neq 0$ this solution is not in $L^2(\mathbb{R})$ because it diverges at infinity. Therefore, besides the trivial solution, there are no classical solutions in $L^2(\mathbb{R})$ (and by the arguments above no weak solutions) of equation 2.1. $\square$

Our next goal is to determine $\sigma_p(\overline{H_l})$. In order to do this, we proceed in the following steps:

1. Show that $0 \in \sigma_p(\overline{a_l})$.
2. Examine the relation between eigenvalues of $\overline{A_l}$ and eigenvalues of $\overline{B_l}$.
3. Conclude that $\overline{H_l}$ has the same eigenvalues as $\overline{H_{l-1}}$ and one additional more.

The first step is easily done:

Corollary 2.1.6. *Let* $l \in \mathbb{N}$ *and* $\psi_0^l(x) := \cosh^{-l}(x)$. *Then* $\psi_0^l \in D(\overline{a_l}) \cap L^1(\mathbb{R})$ *and* $\overline{a_l}\psi_0^l = 0$. *More than that,* $\ker(\overline{a_l}) = \mathbb{C}\psi_0^l$.

Proof. Because of

$$\cosh(x)^{-l} = \frac{2^l}{(e^x + e^{-x})^l} \leq \frac{2^l}{e^{lx} + e^{-lx}} \leq 2^l(e^{-lx}\chi_{\mathbb{R}_+} + e^{lx}\chi_{R_-})$$

we have $\psi_0^l \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. ψ_0^l has the classical derivative $\left(\psi_0^l\right)' = -l \sinh \cosh^{-l-1} = -l \tanh \psi_0^l$. This, together with $\tanh \in L^\infty(\mathbb{R})$, implies $\left(\psi_0^l\right)' \in L^2(\mathbb{R})$. Therefore, $\psi_0^l \in H^{1,2}(\mathbb{R}) = D(\overline{a_l})$. Application of $\overline{a_l}$ on ψ_0^l yields

$$\overline{a_l}\psi_0^l = -i\left(\psi_0^l\right)' - il \tanh \psi_0^l = -i\left(-l \tanh \psi_0^l\right) - il \tanh \psi_0^l = 0$$

and hence $\mathbb{C}\psi_0^l \subseteq \ker(\overline{a_l})$. It remains to prove that $\dim_{\mathbb{C}}(\ker(\overline{a_l})) \leq 1$. Following the same line of reasoning as in the proof of Lemma 2.1.5, we find that each solution $f \in H^{1,2}(\mathbb{R})$ of the equation $\overline{P}f - il \tanh f = 0$ is of the form $f(x) = f(0)\exp\left(-l \int_0^x \tanh(x)\, dx\right)$ for some $f(0) \in \mathbb{C}$. Therefore, $\dim_{\mathbb{C}}(\ker(\overline{a_l})) \leq 1$. $\square$

As for how eigenvalues of $\overline{A_l}$ relate to eigenvalues of $\overline{B_l}$, the following corollary provides an answer.

Corollary 2.1.7. *The following statements hold ($l \in \mathbb{N}$):*

1. *If $\lambda \in \sigma_p(\overline{B_l})$ with eigenvector $\phi \in D(\overline{B_l})$ then $a_l^*\phi \in D(\overline{A_l})$ and $\lambda \in \sigma_p(\overline{A_l})$ with eigenvector $a_l^*\phi$.*

2. *If $\lambda \in \sigma_p(\overline{A_l})$ with eigenvector $\phi \in D(\overline{A_l})$ then $\overline{a_l}\phi \in D(\overline{B_l})$ and if $\overline{a_l}\phi \neq 0$ then $\lambda \in \sigma_p(\overline{B_l})$ with eigenvector $\overline{a_l}\phi$.*

Proof. 1. Let $0 \neq \phi \in D(\overline{B_l}) = H^{2,2}(\mathbb{R})$ be such that $\overline{B_l}\phi = \lambda\phi$. According to Corollary 2.1.4 we have

$$\overline{B_l}\phi = (\overline{H_{l-1}} + l^2)\phi = \lambda\phi$$
$$\iff (\lambda - l^2 - V_{l-1})\phi = \overline{H_0}\phi.$$

Since $v_{l-1} \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we have $V_{l-1}\phi \in H^{2,2}(\mathbb{R})$ and therefore, the left-hand side is in $H^{2,2}(\mathbb{R})$. Thus, the right hand has to be in $H^{2,2}(\mathbb{R})$ too and we can conclude $\phi \in H^{4,2}(\mathbb{R})$. In particular, we obtain $\overline{P}\phi \in H^{3,2}(\mathbb{R}) \subseteq D(\overline{A_l})$. Since $\tanh \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we also have $M_{th}\phi \in D(\overline{A_l})$. Hence, $a_l^*\phi = \overline{P}\phi + ilM_{th}\phi \in D(\overline{A_l})$. According to Lemma 2.1.5, $a_l^*\phi \neq 0$ and we can verify that $a_l^*\phi$ is an eigenvector of $\overline{A_l}$ (by using Corollary 2.1.4):

$$\overline{A_l}a_l^*\phi = a_l^*\overline{a_l}a_l^*\phi = a_l^*\overline{B_l}\phi = \lambda a_l^*\phi$$

2. Let $0 \neq \phi \in D(\overline{A_l}) = H^{2,2}(\mathbb{R})$ such that $\overline{A_l}\phi = (\overline{H_l} + l^2)\phi = \lambda\phi$. We can use the same argument as in 1. to see that $\phi \in H^{4,2}(\mathbb{R})$. Consequently, we obtain $\overline{a_l}\phi \in H^{2,2}(\mathbb{R}) = D(\overline{B_l})$. Now suppose $\overline{a_l}\phi \neq 0$. Then $\overline{a_l}\phi$ is an eigenvector of $\overline{B_l}$ with eigenvalue λ:

$$\overline{B_l}\overline{a_l}\phi = \overline{a_l}a_l^*\overline{a_l}\phi = \overline{a_l}\overline{A_l}\phi = \lambda\overline{a_l}\phi$$

□

Thanks to Corollary 2.1.4 we can rewrite Corollary 2.1.7 in terms of the operators $\overline{H_l}$ and $\overline{H_{l-1}}$.

Corollary 2.1.8. *The following statements hold ($l \in \mathbb{N}$):*

1. *If $\lambda \in \sigma_p(\overline{H_{l-1}})$ with eigenvector $\phi \in D(\overline{H_{l-1}})$ then $a_l^*\phi \in D(\overline{H_l})$ and $\lambda \in \sigma_p(\overline{H_l})$ with eigenvector $a_l^*\phi$.*

2. *If $\lambda \in \sigma_p(\overline{H_l})$ with eigenvector $\phi \in D(\overline{H_l})$ then $\overline{a_l}\phi \in D(\overline{H_{l-1}})$ and if $\overline{a_l}\phi \neq 0$ then $\lambda \in \sigma_p(\overline{H_{l-1}})$ with eigenvector $\overline{a_l}\phi$.*

Proof. This is a direct consequence of Corollary 2.1.4 and Corollary 2.1.7. □

Now we are ready two determine the full point spectrum of $\overline{H_l}$.

Theorem 2.1.9. *Let $l \in \mathbb{N}$. Then*

$$\sigma_p(\overline{H_l}) = \sigma_p(\overline{H_{l-1}}) \cup \{-l^2\}$$

and therefore

$$\sigma_p(\overline{H_l}) = \bigcup_{j=1}^{l} \{-j^2\}.$$

Furthermore, $\dim_{\mathbb{C}}(\ker(\overline{H_l} - \lambda)) = 1$ *for each* $\lambda \in \sigma_p(\overline{H_l})$.

Proof. Let $\lambda \in \sigma_p(\overline{H_l})$ and $0 \neq \phi \in D(\overline{H_l})$ such that $\overline{H_l}\phi = \lambda\phi$. Assume $\overline{a_l}\phi \neq 0$. According to Corollary 2.1.8 we then have $\lambda \in \sigma_p(\overline{H_{l-1}})$. If instead $\overline{a_l}\phi = 0$ then there exists a $c \in \mathbb{C}$ such that $\phi = c\psi_0^l$ (Corollary 2.1.6). Thus, by Corollary 2.1.4, $(\overline{H_l} + l^2)\phi = ca_l^*\overline{a_l}\psi_0^l = 0$ and hence $\lambda = -l^2$. This proves the inclusion "$\subseteq$".

Conversely, let $\lambda \in \sigma_p(\overline{H_{l-1}}) \cup \{-l^2\}$. If $\lambda = -l^2$ we are done since ψ_0^l is an eigenvector of $\overline{H_l}$ with eigenvalue $-l^2$. If $\lambda \in \sigma_p(\overline{H_{l-1}})$ with corresponding eigenvector $\phi \in D(H_{l-1}) = D(H_l)$ then $a_l^*\phi$ is an eigenvector of $\overline{H_l}$ with eigenvalue λ according to Corollary 2.1.8. This proves the other inclusion "$\supseteq$".

Knowing that $\sigma_p(\overline{H_0}) = \emptyset$ (Corollary 1.1.7) we obtain

$$\sigma_p(\overline{H_l}) = \bigcup_{j=1}^{l}\{-j^2\}.$$

Next we prove $\dim_{\mathbb{C}}(\ker(\overline{H_l} - \lambda)) = 1$ (for each $\lambda \in \sigma_p(\overline{H_l})$): Observe that for $\lambda = -l^2 \in \sigma_p(\overline{H_l})$ the claim follows directly from the above results: Thanks to Corollary 2.1.4 we have $\ker(\overline{H_l} + l^2) = \ker(a_l^*\overline{a_l})$ and since $\ker(a_l^*) = \{0\}$ (Lemma 2.1.5), $\ker(\overline{H_l}+l^2) = \ker(\overline{a_l}) = \mathbb{C}\psi_0^l$ by Corollary 2.1.6 so that $\dim_{\mathbb{C}}(\ker(\overline{H_l}+l^2)) = 1$. Since $\sigma_p(\overline{H_1}) = \{-1^2\}$, the claim is true for $l = 1$. Now suppose that the claim is true for one $l \in \mathbb{N}$. Let $\lambda \in \sigma_p(\overline{H_{l+1}})$. We have to show that $\dim_{\mathbb{C}}(\ker(\overline{H_{l+1}} - \lambda)) = 1$. If $\lambda = -(l+1)^2$ we are done by the above reasoning. Assume $\lambda \neq -(l+1)^2$ and choose $\phi_1, \phi_2 \in \ker(\overline{H_{l+1}} - \lambda)$. Since $\lambda \neq -(l+1)^2$, we have $\overline{a_{l+1}}\phi_{1,2} \neq 0$, and we can apply Corollary 2.1.8 to conclude that $\overline{a_{l+1}}\phi_{1,2} \in \ker(\overline{H_l} - \lambda)$. By the induction hypothesis there exists a $c \in \mathbb{C}$ such that $\overline{a_{l+1}}\phi_1 = c\overline{a_{l+1}}\phi_2$ or equivalently $\phi_1 - c\phi_2 \in \ker(\overline{a_{l+1}})$. Thanks to Corollary 2.1.6 we can conclude that there exists a $d \in \mathbb{C}$ such that $\phi_1 - c\phi_2 = d\psi_0^{l+1}$. We obtain

$$-(l+1)^2 d\psi_0^{l+1} = d\overline{H_{l+1}\psi_0^{l+1}} = \overline{H_{l+1}}(\phi_1 - c\phi_2) = \lambda((\phi_1 - c\phi_2)) = \lambda d\psi_0^{l+1}.$$

Therefore, if $d \neq 0$, we find $\lambda = -(l+1)^2$ contradicting the assumption. Hence, $d = 0$ and $\phi_1 = c\phi_2$. □

So far we have seen that $\sigma(\overline{H_0}) \cup \sigma_p(\overline{H_l}) \subseteq \sigma(\overline{H_l})$. Proving equality requires a bit more work and will be done in the following section.

2.2 Calculating the Spectral Measures

The aim of this section is to find out an explicit formula for the spectral measures of $\overline{H_l}$. We will start with a bit of physical intuition to get an idea of what these measures could look like. A small word of warning: The following paragraph is intended to serve only as an intuition and is not meant to be precise or rigourus. Without usage of what is called "Gelfand triples", most of the following expressions are ill-defined. A good starting point for learning more about Gelfand triples might be [4].

Given a Hamiltonian H, physicists like to think of the spectral decomposition as an integral of the form

$$H = \int_{\mathbb{R}} \lambda |f_\lambda\rangle\langle f_\lambda|\, d\lambda, \tag{2.2}$$

where $|f_\lambda\rangle$ denotes an eigenstate of the Hamiltonian with eigenvalue λ: $H|f_\lambda\rangle = \lambda|f_\lambda\rangle$. Notice, that in physics language this does not mean, that $|f_\lambda\rangle$ has to be in

$L^2(\mathbb{R})$ (nonetheless they are called eigenfunctions). Furthermore, an expression like $f(H)$ for a function f can be thought of as

$$f(H) = \int_{\mathbb{R}} f(\lambda)|f_\lambda\rangle\langle f_\lambda|\, d\lambda. \tag{2.3}$$

The spectral measure $E_S = \chi_S(H)$ would then be something like

$$\chi_S(H) = \int_S |f_\lambda\rangle\langle f_\lambda|\, d\lambda. \tag{2.4}$$

For the special case at hand, we know, that given a solution $|f_\lambda^{l-1}\rangle$ of $H_{l-1}|f_\lambda^{l-1}\rangle = \lambda|f_\lambda^{l-1}\rangle$ we get an eigenvector of H_l by applying a_l^*: $H_l|a_l^* f_\lambda^{l-1}\rangle = \lambda|a_l^* f_\lambda^{l-1}\rangle$ (Corollary 2.1.8). In the language of bras and kets we furthermore have the relations $|a_l^* f_\lambda^{l-1}\rangle = a_l^*|f_\lambda^{l-1}\rangle$ and $\langle a_l^* f_\lambda^{l-1}| = \langle f_\lambda^{l-1}|a_l$. Applying this to equation 2.4 and interchanging integration and a_l^*, a_l leads to

$$\chi_S(H_l) = a_l^* \left(\int_S |f_\lambda^{l-1}\rangle\langle f_\lambda^{l-1}|\, d\lambda \right) a_l = a_l^* \chi_S(H_{l-1}) a_l. \tag{2.5}$$

Hence, the spectral projections of H_l are completely determined by those of H_{l-1}. As we already know them for $l = 0$, this would be enough to deduce them for arbitrary l.

Now that we have a rough idea, we need to make this idea more formal. Let $E^l : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R}))$ be the spectral measure of $\overline{H_l}$ and for $S \in \mathcal{B}(\mathbb{R})$ we write $E_S^l \coloneqq E^l(S)$ as always. To get a bit more intuition, let us start with the simplest case by setting $S \coloneqq \{-j^2\} \in \sigma_p(\overline{H_l})$ ($j \in \{1, 2, \ldots, l\}$). Denoting with $\psi_j \in D(\overline{H_l})$ the normalized eigenvector with eigenvalue $-j^2$ of $\overline{H_l}$ (there is exactly one according to Theorem 2.1.9) and $E_j^l \coloneqq E_{\{-j^2\}}^l$ we have

$$E_j^l f = \langle \psi_j, f\rangle \psi_j \quad \text{for } f \in L^2(\mathbb{R}) \tag{2.6}$$

and as $\psi_j \in D(\overline{H_l}) \subseteq D(a_l^*)$ the operator $a_l^* E_j^{l-1} a_l$ is given by

$$a_l^* E_j^{l-1} a_l f = \langle \psi_j, a_l f\rangle a_l^* \psi_j = \langle a_l^* \psi_j, f\rangle a_l^* \psi_j \quad \text{for } f \in \mathscr{S}(\mathbb{R}). \tag{2.7}$$

Therefore, we can observe that $a_l^* E_j^l a_l$ (defined on $D(a_l) = \mathscr{S}(\mathbb{R})$) extends to the bounded operator $L^2(\mathbb{R}) \ni f \mapsto \langle a_l^* \psi_j, f\rangle a_l^* \psi_j$. However, to make this a projection, we have to introduce a scaling factor:

Lemma 2.2.1. *Let $l \in \mathbb{N}$, $-j^2 \in \sigma_p(\overline{H_{l-1}})$ ($j = \{1, 2, \ldots, l-1\}$), $\psi_j \in D(\overline{H_{l-1}})$ the corresponding normalized eigenvector and denote with $E^k : \mathcal{B}(\mathbb{R}) \to L^2(\mathbb{R})$ the spectral measure of $\overline{H_k}$ for $k \in \mathbb{N}$. Then the operator*

$$E_j^l : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad E_j^l f = (-j^2 + l^2)^{-1} \langle a_l^* \psi_j, f\rangle a_l^* \psi_j$$

extends $(-j^2 + l^2)^{-1} a_l^* E_{\{-j^2\}}^{l-1} a_l : \mathscr{S}(\mathbb{R}) \to L^2(\mathbb{R})$ boundedly. Furthermore, E_j^l is an orthogonal projection and we have $E_j^l = E_{\{-j^2\}}^l$.

Proof. That E_j^l is a bounded extension has already been proven. For $f, g \in L^2(\mathbb{R})$ we calculate

$$\begin{aligned} \langle f, E_j^l g\rangle &= (-j^2 + l^2)^{-1} \langle a_l^* \psi_j, g\rangle \langle f, a_l^* \psi_j\rangle \\ &= (-j^2 + l^2)^{-1} \langle \langle a_l^* \psi_j, f\rangle a_l^* \psi_j, g\rangle = \langle E_j^l f, g\rangle \end{aligned}$$

and hence $E_j^l = \left(E_j^l\right)^*$. Now, observe that

$$\|a_l^*\psi_j\|_{2,\mathbb{R}}^2 = \langle \overline{a}_l a_l^*\psi_j, \psi_j\rangle = \langle (\overline{H_{l-1}} + l^2)\psi_j, \psi_j\rangle = (-j^2 + l^2),$$

where we used $\|\psi_j\|_{2,\mathbb{R}} = 1$ and $\psi_j \in D(\overline{H_{l-1}}) = H^{2,2}(\mathbb{R})$ and therefore $a_l^*\psi_j \in H^{1,2}(\mathbb{R}) = D(\overline{a_l}) = D(a_l^{**})$ (by Corollary 2.1.1 a_l is closable). Now, it follows from a direct calculation that E_j^l is a projection. Indeed, for $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned}\left(E_j^l\right)^2 f &= (-j^2 + l^2)^{-1}\langle a_l^*\psi_j, E_j^l f\rangle a_l^*\psi_j \\ &= (-j^2 + l^2)^{-2}\|a_l^*\psi_j\|_{2,\mathbb{R}}^2\langle a_l^*\psi_j, f\rangle a_l^*\psi_j = E_j^l f.\end{aligned}$$

By design, we have $E_j^l = \mathrm{Proj}_{\mathrm{span}(a_l^*\psi_j)}$ and since $a_l^*\psi_j$ is an eigenvector of $\overline{H}_l$ with eigenvalue $-j^2$ (Corollary 2.1.8) and $\ker(\overline{H}_l + j^2)$ is one-dimensional (Theorem 2.1.9), this leads to $E_j^l = \mathrm{Proj}_{\ker(\overline{H}_l + j^2)} = E_{\{-j^2\}}^l$. □

Our physical reasoning from above indeed leads to the correct relation between E_S^l and E_S^{l-1}, at least for $S \subseteq \sigma_p(\overline{H_{l-1}})$. Next, we can try to generalize the result to arbitrary $S \in \mathcal{B}(\mathbb{R})$. Observe that

$$(-j^2 + l^2)^{-1}E_{\{-j^2\}}^{l-1} = R^{l-1}(-l^2)E_{\{-j^2\}}^{l-1}, \tag{2.8}$$

where $R^{l-1}(-l^2) \coloneqq (\overline{H_{l-1}} + l^2)^{-1}$ denotes the resolvent. Our hope is that for arbitrary $S \in \mathcal{B}(\mathbb{R})$ we find something like

$$E_S^l = a_l^* R^{l-1}(-l^2)E_S^{l-1}a_l. \tag{2.9}$$

Of course, this can certainly not be true, as $\overline{H}_l$ has one more eigenvalue than $\overline{H_{l-1}}$ and therefore E^{l-1} does not have all the information that is needed for E^l. We will see later on that we can overcome this issue by adding a projection onto the eigenspace of the additional eigenvalue. Leaving this aside, without consideration of rigorous details we can rewrite equation 2.9 as follows

$$E_S^l = \frac{a_l^*}{\sqrt{\overline{H_{l-1}} + l^2}} E_S^{l-1} \frac{a_l}{\sqrt{\overline{H_{l-1}} + l^2}} \tag{2.10}$$

$$= \frac{a_l^*}{\sqrt{\overline{a}_l a_l^*}} E_S^{l-1} \frac{a_l}{\sqrt{\overline{a}_l a_l^*}}. \tag{2.11}$$

Recall that for bounded operators $T \in B(H)$ we can use the polar decomposition to write $T = U\sqrt{T^*T}$ with $U \in B(H)$ being a partial isometry. If $\sqrt{T^*T}$ was invertible, then $U = T\sqrt{T^*T}^{-1}$ which reminds us of the terms in equation 2.11. Again, we have to formalize these ideas. Recall that the polar decomposition can be generalized to closed densely defined operators.

Theorem 2.2.2 ([9], page 138)**.** *Let $T : H \supseteq D(T) \to H$ be a closed densely defined linear operator. Then T^*T is a positive self-adjoint operator and there exists a partial isometry $U \in B(H)$ with initial space $\ker(U)^\perp = \ker(T)^\perp = \overline{\mathrm{Ran}(T^*)}$ and final space $\mathrm{Ran}(U) = \overline{\mathrm{Ran}(T)} = \ker(T^*)^\perp$ such that*

$$T = U|T|,$$

where $|T| \coloneqq (T^*T)^{\frac{1}{2}}$ denotes the modulus of T. Furthermore, the following relations hold:

1. $UU^* = \mathrm{Proj}_{\ker(T^*)^\perp}$
2. $T^* = |T|U^*$

Equation 2.11 seems to suggest a relation between E_S^l, E_S^{l-1} and the polar decomposition of a_l^*. As we will need the partial isometry later on, let us start by giving it a name:

Definition 2.2.1. For $l \in \mathbb{N}$ we denote with $U_l \in B(L^2(\mathbb{R}))$ the partial isometry from the polar decomposition (Theorem 2.2.2) of the closed operator a_l^*.

Now, we can quickly prove two important facts. First, some properties of U_l and second, how U_l relates $\overline{H_l}$ with $\overline{H_{l-1}}$.

Lemma 2.2.3. *Let $l \in \mathbb{N}$. Then U_l is an isometry. Furthermore, restricted to $D(\overline{a_l})$, U_l^* is given by*

$$U_l^*\big|_{D(\overline{a_l})} = \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} \left(\overline{P} - ilM_{th}\right). \tag{2.12}$$

In particular, for $l = 1$, one obtains

$$U_1^* = \frac{\overline{P}}{(\overline{P}^2 + 1)^{\frac{1}{2}}} - i(\overline{P}^2 + 1)^{-\frac{1}{2}} M_{th}. \tag{2.13}$$

Proof. Since $\ker(a_l^*) = \{0\}$ (Lemma 2.1.5), U_l is isometric. As $-l^2 \notin \sigma(\overline{H_{l-1}})$ the operator $|a_l^*| = \sqrt{\overline{a_l} a_l^*} = \sqrt{\overline{H_{l-1}} + l^2}$ has the bounded inverse $\left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}}$. Theorem 2.2.2 yields

$$a_l^{**} = \overline{a_l} = |a_l^*| U_l^* \implies U_l^*\big|_{D(\overline{a_l})} = \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} \overline{a_l},$$

which, together with Corollary 2.1.1, proves equation 2.12. In the case $l = 1$, it is easy to see how equation 2.12 simplifies to equation 2.13: Because of $\overline{H_0} = \overline{P}^2$ and the boundedness of the function $\mathbb{R} \ni x \mapsto x(x^2+1)^{-\frac{1}{2}}$, we can drop the above restriction to $D(\overline{a_1})$ and obtain equation 2.13. □

Lemma 2.2.4. *Let $l \in \mathbb{N}$. The operators $\overline{H_l}$ and $\overline{H_{l-1}}$ are related via*

$$\overline{H_l} = U_l \overline{H_{l-1}} U_l^* - l^2 \mathrm{Proj}_{\ker(\overline{a_l})}.$$

Proof. We find

$$\begin{aligned}
\overline{H_l} &= a_l^* \overline{a_l} - l^2 \quad \text{(Corollary 2.1.4)} \\
&= U_l |a_l^*| |a_l^*| U_l^* - l^2 \quad \text{(Theorem 2.2.2)} \\
&= U_l \overline{a_l} a_l^* U_l^* - l^2 \\
&= U_l (\overline{H_{l-1}} + l^2) U_l^* - l^2 \quad \text{(Corollary 2.1.4)} \\
&= U_l \overline{H_{l-1}} U_l^* - l^2 (1 - U_l U_l^*).
\end{aligned}$$

In view of Theorem 2.2.2, we have $1 - U_l U_l^* = 1 - \mathrm{Proj}_{\ker(\overline{a_l})^\perp} = \mathrm{Proj}_{\ker(\overline{a_l})}$, which proves the claim. □

We can now hope that the relation between $\overline{H_l}$ and $\overline{H_{l-1}}$ carries over to the relation between the corresponding spectral measures. And indeed, we have the following Theorem.

Theorem 2.2.5. *Let $l \in \mathbb{N}$ and E^{l-1} be the spectral measure of $\overline{H_{l-1}}$. Then the spectral measure E^l of $\overline{H_l}$ is given by*

$$E^l_S = U_l E^{l-1}_S U_l^* + \delta_{-l^2}(S)\mathrm{Proj}_{\ker(\overline{H_l}+l^2)} \quad \forall S \in \mathcal{B}(\mathbb{R}),$$

where δ_{-l^2} denotes the Dirac measure centered at the point $-l^2$.

Proof. Recall that $\ker(\overline{H_l} + l^2) = \ker(\overline{a_l})$. At first, we have to make sure that for $S \in \mathcal{B}(\mathbb{R})$ the operator

$$E_S := U_l E^{l-1}_S U_l^* + \delta_{-l^2}(S)\mathrm{Proj}_{\ker(\overline{a_l})}$$

is an orthogonal projection and that $E : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R})), S \mapsto E_S$ defines a projection valued measure.

1. Because U_l is isometric ($U_l^* U_l = 1$), $E_S^* = E_S$. Since $\mathrm{Ran}(U_l) = \ker(\overline{a_l})^\perp$, we find $\mathrm{Proj}_{\ker(\overline{a_l})} U_l = 0$ as well as $U_l^* \mathrm{Proj}_{\ker(\overline{a_l})} = 0$. Therefore, $(E_S)^2 = E_S$. Hence, E_S is an orthogonal projection.

2. Clearly, $E_\emptyset = 0$ and
$$E_{\mathbb{R}} = U_l U_l^* + \mathrm{Proj}_{\ker(\overline{a_l})} = \mathrm{Proj}_{\ker(\overline{a_l})^\perp} + \mathrm{Proj}_{\ker(\overline{a_l})} = 1.$$

3. If $(S_j)_{j\in\mathbb{N}} \subseteq \mathcal{B}(\mathbb{R})$ is a pairwise disjoint family of Borel sets then $\delta_{-l^2}(\cup_{j\in\mathbb{N}} S_j) = \sum_{j\in\mathbb{N}} \delta_{-l^2}(S_j)$ and using that E^{l-1} already is a projection valued measure we obtain $E_{\cup_{j\in\mathbb{N}} S_j} f = \sum_{j\in\mathbb{N}} E_{S_j} f$ for all $f \in L^2(\mathbb{R})$.

We notice that for $f \in L^2(\mathbb{R})$ the measures $\mu^E_f(S) := \langle f, E_S f\rangle$ are given by

$$\begin{aligned}\mu^E_f(S) &= \langle U_l^* f, E^{l-1}_S U_l^* f\rangle + \delta_{-l^2}(S)\langle f, \mathrm{Proj}_{\ker(\overline{a_l})} f\rangle \\ &= \mu^{l-1}_{U_l^* f}(S) + \delta_{-l^2}(S)\langle f, \mathrm{Proj}_{\ker(\overline{a_l})} f\rangle,\end{aligned}$$

where we set $\mu^{l-1}_g := \langle g, E^{l-1}_S g\rangle$ for $g \in L^2(\mathbb{R})$. Define the self-adjoint operator

$$A := \int_{\mathbb{R}} \lambda \, dE_\lambda$$

and use the previous lemma to derive that for $f \in D(\overline{H_l})$

$$\begin{aligned}\langle f, H_l f\rangle &= \langle U_l^* f, \overline{H_{l-1}} U_l^* f\rangle - l^2\langle f, \mathrm{Proj}_{\ker(\overline{a_l})} f\rangle \\ &= \int_{\mathbb{R}} \lambda \, d\mu^{l-1}_{U_l^* f} + \int_{\mathbb{R}} \lambda\langle f, \mathrm{Proj}_{\ker(\overline{a_l})} f\rangle \, d\delta_{-l^2} \\ &= \int_{\mathbb{R}} \lambda \, d\mu^E_f = \langle f, Af\rangle.\end{aligned}$$

By the polarization identity this shows that $\langle g, \overline{H_l} f\rangle = \langle g, Af\rangle$ for all $f, g \in D(\overline{H_l})$ and since $D(\overline{H_l}) \subseteq L^2(\mathbb{R})$ is dense, we have $\overline{H_l} \subset A$. Since $\overline{H_l}$ as well as A are self-adjoint, we even have the equality $A = \overline{H_l}$. By the uniqueness of the spectral measures this implies $E = E^l$. □

Remark 2.2.6. Theorem 2.2.5 provides an answer to whether there are parts of the spectrum below zero which do not belong to the point spectrum. We find

$$\sigma(\overline{H_l}) = \operatorname{supp}(E^l) = \operatorname{supp}(E^{l-1}) \cup \{-l^2\} = \sigma(\overline{H_{l-1}}) \cup \{-l^2\}.$$

As $\sigma(\overline{H_0}) = \mathbb{R}_{\geq 0}$ we find

$$\sigma(\overline{H_l}) = \mathbb{R}_{\geq 0} \cup \left(\bigcup_{j=1}^{l} \{-j^2\} \right).$$

So that there is indeed only point-spectrum below zero.

In the subsequent sections we are mainly interested in operators of the form $e^{it\overline{H_l}}$. Now that the spectral measures are known, we are in a quite good position to relate $e^{it\overline{H_l}}$ and $e^{it\overline{H_{l-1}}}$.

Corollary 2.2.7. *Let $g \in L^\infty(\sigma(\overline{H_l}))$ and $l \in \mathbb{N}$. Then*

$$g(\overline{H_l}) = U_l g(\overline{H_{l-1}}) U_l^* + g(-l^2) \operatorname{Proj}_{\ker(\overline{a_l})}. \tag{2.14}$$

Proof. This is a consequence of Theorem 2.2.5. For $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned}
\langle f, g(\overline{H_l}) f \rangle &= \int_{\mathbb{R}} g(\lambda)\, d\mu_f^l \\
&= \int_{\mathbb{R}} g(\lambda)\, d\mu_{U_l^* f}^{l-1} + \int_{\mathbb{R}} g(\lambda) \langle f, \operatorname{Proj}_{\ker(\overline{a_l})} f \rangle \, d\delta_{-l^2} \\
&= \langle U_l^* f, g(\overline{H_{l-1}}) U_l^* f \rangle + g(-l^2) \langle f, \operatorname{Proj}_{\ker(\overline{a_l})} f \rangle \\
&= \langle f, \left(U_l g(\overline{H_{l-1}}) U_l^* + g(-l^2) \operatorname{Proj}_{\ker(\overline{a_l})} \right) f \rangle.
\end{aligned}$$

The polarization identity now proves the claim. □

With the help of Lemma 2.2.3 we can try to eliminate the U_l operators in equation 2.14. At least for $l = 1$ this is easily done.

Corollary 2.2.8. *Let $g \in L^\infty(\sigma(\overline{H_1}))$ be such that $g(\overline{P}^2)\mathscr{S}(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R})$. Then*

$$g(H_1)\big|_{\mathscr{S}(\mathbb{R})} = \left(\overline{P} + iM_{th}\right) \frac{g(\overline{P^2})}{\overline{P}^2 + 1} \left(\overline{P} - iM_{th}\right) + g(-1) \operatorname{Proj}_{\ker(\overline{a_1})}.$$

Proof. Since $\left(\overline{P}^2 + ^1\right)^{-1} \mathscr{S}(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R})$ and $g(\overline{P}^2)\mathscr{S}(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R})$ the above equation is indeed well-defined on $\mathscr{S}(\mathbb{R})$. Then the Corollary is a direct consequence of Lemma 2.2.3 (equation 2.13) and Corollary 2.2.7. □

Recursively, we obtain the following expression for $g(\overline{H_l})$:

Theorem 2.2.9. *Let $g \in L^\infty(\sigma(\overline{H_l}))$ be such that $g(\overline{P}^2)\mathscr{S}(\mathbb{R}) \subseteq \mathscr{S}(\mathbb{R})$. Then*

$$\begin{aligned}
g(\overline{H_l})\big|_{\mathscr{S}(\mathbb{R})} = &\left(\prod_{j=0}^{l-1} a_{l-j}^* \right) \frac{g(\overline{P}^2)}{\prod_{j=1}^{l} \left(\overline{P}^2 + j^2 \right)} \left(\prod_{j=1}^{l} \overline{a_j} \right) \\
&+ \sum_{j=1}^{l-1} \left[\frac{g(-j^2)}{\prod_{k=j+1}^{l} (-j^2 + k^2)} \left(\prod_{k=0}^{l-(j+1)} a_{l-k}^* \right) \operatorname{Proj}_{\ker(\overline{a_j})} \left(\prod_{k=j+1}^{l} \overline{a_k} \right) \right] \\
&+ g\left(-l^2\right) \operatorname{Proj}_{\ker(\overline{a_l})}.
\end{aligned} \tag{2.15}$$

Proof. At first, we have to make sure that the above expression is well-defined on $\mathscr{S}(\mathbb{R})$. We can use the same reasoning as in the proof of Corollary 2.2.8 to see that the first term is well-defined on $\mathscr{S}(\mathbb{R})$. The third term is also easy to handle. The second term (the one including the projections) is more complicated. Since $\mathrm{Proj}_{\ker(\overline{a_j})} = \langle \psi_0^j, \cdot \rangle \psi_0^j$ (recall the definition of ψ_0^j from Corollary 2.1.6), we have to prove that $a_{j+1}^* \psi_0^j \in D(a_{j+2}^*) = H^{1,2}(\mathbb{R})$, $a_{j+2}^* a_{j+1}^* \psi_0^j \in D(a_{j+3}^*) = H^{1,2}(\mathbb{R})$ and so on. This reduces to whether $\left(\psi_0^j\right)^{(n)} \in H^{1,2}(\mathbb{R})$ for all $n \in \mathbb{N}$, where $\left(\psi_0^j\right)^{(n)}$ denotes the n-th derivative. We claim that there exists a polynomial P_n such that $\left(\psi_0^j\right)^{(n)} = P_n(\tanh)\psi_0^j$. Thanks to the proof of Corollary 2.1.6 we already know, that the claim is true for $n = 1$ (choose $P_1(x) = -jx$). Let the claim be true for one $n \in \mathbb{N}$. Then

$$\begin{aligned}
\left(\psi_0^j\right)^{(n+1)} &= \left(P_n(\tanh)\psi_0^j\right)' \\
&= (\tanh)' P_n'(\tanh)\psi_0^j + P_n(\tanh)\left(\psi_0^j\right)' \\
&= \left((1-\tanh^2)P_n'(\tanh) - j\tanh P_n(\tanh)\right)\psi_0^j.
\end{aligned}$$

Setting $P_{n+1}(x) = (1-x^2)P_n'(x) - jxP_n(x)$, which is again a polynomial, proves the claim. Since tanh is bounded and smooth, $P_n(\tanh)$ is bounded and smooth too. Therefore, $\left(\psi_0^j\right)^{(n)} = P_n(\tanh)\psi_0^j \in H^{1,2}(\mathbb{R})$ because $\psi_0^j \in H^{1,2}(\mathbb{R})$ (Corollary 2.1.6). Hence, the expression in equation 2.15 is indeed well-defined.

We prove equation 2.15 by induction. For $l = 1$ the claim follows from Corollary 2.2.8. Let $l \in \mathbb{N}$ be such that equation 2.15 holds true. Thanks to Corollary 2.2.7 we have

$$g(\overline{H_{l+1}}) = U_{l+1} g(\overline{H_l}) U_{l+1}^* + g(-(l+1)^2)\mathrm{Proj}_{\ker(\overline{a_{l+1}})}.$$

Using Lemma 2.2.3 (equation 2.12) we obtain

$$\begin{aligned}
g(\overline{H_{l+1}})\big|_{\mathscr{S}(\mathbb{R})} &= \left(\overline{P} + i(l+1)M_{th}\right) \frac{g(\overline{H_l})}{\overline{H_l} + (l+1)^2} \left(\overline{P} - i(l+1)M_{th}\right) \\
&\quad + g\left(-(l+1)^2\right)\mathrm{Proj}_{\ker(\overline{a_{l+1}})}.
\end{aligned}$$

Setting $f : \sigma(\overline{H_l}) \to \mathbb{R}, x \mapsto \frac{g(x)}{x+(l+1)^2}$ we can rewrite this equation as

$$g(\overline{H_{l+1}})\big|_{\mathscr{S}(\mathbb{R})} = a_{l+1}^* f(\overline{H_l})\overline{a_{l+1}} + g\left(-(l+1)^2\right)\mathrm{Proj}_{\ker(\overline{a_{l+1}})}.$$

Since f is bounded, we can use the induction hypothesis and get

$$\begin{aligned}
&g(\overline{H_{l+1}})\big|_{\mathscr{S}(\mathbb{R})} = \\
&a_{l+1}^* \left(\prod_{j=0}^{l-1} a_{l-j}^*\right) \frac{f(\overline{P}^2)}{\prod_{j=1}^{l}\left(\overline{P}^2 + j^2\right)} \left(\prod_{j=1}^{l} \overline{a_j}\right) \overline{a_{l+1}} \\
&+ a_{l+1}^* \sum_{j=1}^{l-1} \left[\frac{f\left(-j^2\right)}{\prod_{k=j+1}^{l}\left(-j^2+k^2\right)} \left(\prod_{k=0}^{l-(j+1)} a_{l-k}^*\right) \mathrm{Proj}_{\ker(\overline{a_j})} \left(\prod_{k=j+1}^{l} \overline{a_k}\right)\right] \overline{a_{l+1}} \\
&+ a_{l+1}^* f\left(-l^2\right)\mathrm{Proj}_{\ker(\overline{a_l})}\overline{a_{l+1}} + g\left(-(l+1)^2\right)\mathrm{Proj}_{\ker(\overline{a_{l+1}})}.
\end{aligned}$$

Therefore, using the definition of f and reorganizing the indices, we obtain

$$
\begin{aligned}
& g(\overline{H_{l+1}})\big|_{\mathscr{S}(\mathbb{R})} = \\
& \left(\prod_{j=0}^{(l+1)-1} a^*_{(l+1)-j}\right) \frac{g(\overline{P}^2)}{\prod_{j=1}^{l+1}\left(\overline{P}^2 + j^2\right)} \left(\prod_{j=1}^{l+1} \overline{a_j}\right) \\
& + \sum_{j=1}^{(l+1)-1} \left[\frac{g\left(-j^2\right)}{\prod_{k=j+1}^{l+1}\left(-j^2 + k^2\right)} \left(\prod_{k=0}^{(l+1)-(j+1)} a^*_{(l+1)-k}\right) \operatorname{Proj}_{\ker(\overline{a_j})} \left(\prod_{k=j+1}^{l+1} \overline{a_k}\right)\right] \\
& + g\left(-(l+1)^2\right) \operatorname{Proj}_{\ker(\overline{a_{l+1}})}.
\end{aligned}
$$

□

Chapter 3

Scattering Theory and Transparent Potentials

Now that we have successfully worked out the spectral measures of our Hamiltonian, we are in a good position to start understanding the scattering behavior of the quantum system described by the Pöschl-Teller potential. The first part of this chapter deals with the most fundamental theoretical background of scattering theory and how the purely mathematical definitions can be understood from the viewpoint of a physicist. This chapter is intended to give the important definitions we need in the following sections and is certainly not an introduction to scattering theory. A much more thorough description of scattering theory can be found, for example, in [11]. The second part applies our results from the previous sections to the case at hand.

3.1 Wave Operators, Scattering Operators and Their Physical Meaning

Let $A : \mathscr{H} \supseteq D(A) \to \mathscr{H}$ be a self-adjoint operator, $\phi_0 \in D(A)$, $\phi : \mathbb{R} \to \mathscr{H}$ differentiable and $\phi(\mathbb{R}) \subseteq D(A)$. Then the initial value problem

$$i\phi'(t) = A\phi(t) \quad \forall t \in \mathbb{R}, \quad \phi(0) = \phi_0 \tag{3.1}$$

is called Schrödinger equation and the operator A is called Hamiltonian. One can show that a unique solution of 3.1 exists and is given by $\phi(t) = e^{-itA}\phi_0$ ([11]). One says that the operator e^{-itA} describes the time evolution of a state ϕ_0. In physics, the Hamiltonian (usually called H) is often of the form $H = -\Delta + V$ where V is a multiplication operator with some function v called the potential. For bounded v and the one-dimensional Laplacian we have seen that H is indeed self-adjoint. One goal of scattering theory is to compare two different time evolutions e^{-itA} and e^{-itB}. Therefore, one defines the wave operators $\Omega_\pm$.

Definition 3.1.1. Let $A : \mathscr{H} \supseteq D(A) \to \mathscr{H}$ and $B : \mathscr{H} \supseteq D(B) \to \mathscr{H}$ be two self-adjoint operators. Set $P_A^{ac} \in B(\mathscr{H})$ to be the projection onto the absolute continuous subspace $\mathscr{H}_{ac}^A := \{x \in \mathscr{H} : \mu_x^A \text{ is absolutely continuous with respect to } \lambda\}$. If the limits

$$\Omega_\pm := \operatorname*{s-lim}_{t\to\pm\infty} e^{itB} e^{-itA} P_A^{ac}$$

exist then $\Omega_\pm$ are called *wave operators (Møller operators)* of the pair (A, B).

In many applications in physics we set $A = -\Delta$ and $B = H = -\Delta + V$ and interpret A as describing the evolution of a free state which does not interact with any potential, whereas B describes the interaction of a state with a given potential V. Assuming the wave operators exist, we define the scattering operator S.

Definition 3.1.2. Let (A, B) be a pair of self-adjoint operators for which both wave operators Ω_+ and Ω_- exist. Then the *scattering operator* S is defined as

$$S := \Omega_+^* \Omega_-.$$

Just a few words about the physical meaning of the operators $\Omega_\pm$ and S. Take Figure 3.1 as an illustration of the following explanation (see also [11] and [10]). If the effect of the potential is not too far-reaching, then we can expect the motion of the scattered particle (gray line in Figure 3.1) to be more or less free if the particle is far away from the potentials center. Suppose that we were give an outgoing respectively incoming state $\psi_\pm$ (which lie in $\mathscr{H}_{ac}^A$), then we would expect to find a state ψ such that $\|e^{-itB}\psi - e^{-itA}\psi_\pm\|_{2,\mathbb{R}} \to 0$ as $t \to \pm\infty$. By the unitarity of e^{-itB} we have $\|e^{-itB}\psi - e^{-itA}\psi_\pm\|_{2,\mathbb{R}} = \|\psi - e^{itB}e^{-itA}\psi_\pm\|_{2,\mathbb{R}}$ and therefore, we see that $\psi = \Omega_\pm\psi_\pm$. That is why we say that the wave operators relate the free and the interacting dynamics. Now, the scattering operator takes the asymptotic free incoming state ψ_- (represented by the orange line) and maps it to the asymptotic free outgoing state ψ_+ (represented by the dotted green line).

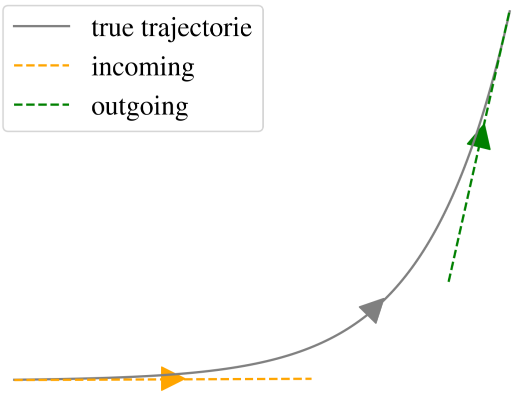

Figure 3.1: Illustration of incoming and outgoing waves.

In the next section we set $A = \overline{H_0}$ and $B = \overline{H_l}$ and want to determine the wave operators and the scattering operator of the pair $(\overline{H_0}, \overline{H_l})$. In that case, Definition 3.1.1 simplifies since $P_{H_0}^{ac} = 1$ according to Corollary 1.1.7. We introduce the following notation:

Definition 3.1.3. Let $l \in \mathbb{N} \cup \{0\}$. We set $\Omega_\pm^l$ and S_l to be the wave operators and scattering operators of the pair $(\overline{H_0}, \overline{H_l})$.

Based on the scattering operator, we can now say, what a transparent potential is. To do so, we first introduce the notion of moving directions of a wave. Let $\psi \in L^2(\mathbb{R})$ be any wave function. We call the wave

1. right-moving if $\text{supp}(\mathcal{F}\psi) \subseteq \mathbb{R}_+$,

2. left-moving if $\text{supp}(\mathcal{F}\psi) \subseteq \mathbb{R}_-$.

Observe that we can split any wave function ψ in a right moving part $\psi_+ := E^{\overline{P}}_{\mathbb{R}_+}\psi$ and a left moving part $\psi_- := E^{\overline{P}}_{\mathbb{R}_-}\psi$. Let ψ be any incoming state and $\tilde{\psi} := S\psi$ the corresponding outgoing state. We are interested in whether left-moving waves stay left-moving waves (i.e. $\tilde{\psi}_-$ is completely determined by ψ_-) and whether right-moving waves stay right moving waves (i.e. $\tilde{\psi}_+$ is completely determined by ψ_+). In this case we speak of transparency. In the case where parts of left-moving waves get mapped to right-moving ones and vice versa we speak of reflectivity.

3.2 Transparency of the Pöschl-Teller Potential

We start by analyzing the free dynamics (see also [11], page 169).

Theorem 3.2.1. *Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the interaction-free time evolution is given by*

$$\left(e^{-it\overline{H_0}} f\right)(x) = \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}} e^{i\frac{(x-y)^2}{4t}} f(y)\, d\lambda(y)$$

for $t \in \mathbb{R} \setminus \{0\}$.

Proof. Set $e_t(p) := e^{-itp^2}$. Then $f_t := e^{-it\overline{H_0}} f = \mathcal{F}^{-1}\left(e_t \mathcal{F}(f)\right)$. Let $n \in \mathbb{N}$ and set $e_t^n(p) := e^{-\left(it+\frac{1}{n}\right)p^2}$. We have

1. $e_t^n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$
2. $e_t^n \to e_t$ pointwise as $n \to \infty$
3. $\sup_{n\in\mathbb{N}} \|e_t^n\|_{\infty,\mathbb{R}} = 1 < \infty$

Therefore, $e_t^n(\overline{P}) \xrightarrow{SOT} e_t(\overline{P})$ for $n \to \infty$. Furthermore, the Fourier transform of e_t^n exists (since $e_t^n \in L^2(\mathbb{R})$ as opposed to e_t) and hence

$$f_t = e_t(\overline{P}) f \underset{n\to\infty}{\longleftarrow} f_t^n := e_t^n(\overline{P}) f = \mathcal{F}^{-1}\left(e_t^n \mathcal{F}(f)\right) = (2\pi)^{-\frac{1}{2}} \mathcal{F}^{-1}\left(e_t^n\right) * f.$$

Since $(e_t^n)' = -2\mathrm{id}_{\mathbb{R}}\left(it + \frac{1}{n}\right) e_t^n \in L^2(\mathbb{R})$ we find

$$\begin{aligned} 0 &= \mathcal{F}^{-1}\left((e_t^n)'\right)(x) + \mathcal{F}^{-1}\left(2\mathrm{id}_{\mathbb{R}}\left(it + \frac{1}{n}\right) e_t^n\right)(x) \\ \iff 0 &= -ix\mathcal{F}^{-1}\left(e_t^n\right)(x) - 2\left(it + \frac{1}{n}\right) i \left(\mathcal{F}^{-1}\left(e_t^n\right)\right)'(x) \\ \iff 0 &= \left(\mathcal{F}^{-1}\left(e_t^n\right)\right)'(x) + \frac{nx}{2(int+1)} \mathcal{F}^{-1}\left(e_t^n\right)(x). \end{aligned}$$

We can solve this ordinary differential equation and conclude that there exists a constant $C \in \mathbb{C}$ such that $\mathcal{F}^{-1}\left(e_t^n\right)(x) = C \exp\left(-\frac{n}{4(int+1)} x^2\right)$. The constant C is determined via

$$C = \mathcal{F}^{-1}\left(e_t^n\right)(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e_t^n(x) dx = \sqrt{\frac{n}{2(int+1)}}.$$

In total we get

$$
\begin{aligned}
f_t^n(x) &= \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left(e_t^n\right) * f(x) = \sqrt{\frac{n}{4\pi(int+1)}} \int_{\mathbb{R}} e^{-\frac{n(x-y)^2}{4(int+1)}} f(y)\, dy \\
&\xrightarrow[n\to\infty]{} \sqrt{\frac{1}{4\pi it}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4it}} f(y)\, dy =: \tilde{f}_t(x),
\end{aligned}
$$

where we used Lebesgue Dominated Convergence (with majorant $|f|$). But since also $f_t^n \to f_t$ in norm, we have $f_t = \tilde{f}_t$. □

Using this explicit form, we can immediately draw a physical conclusion.

Lemma 3.2.2. *Let $f_0 \in L^2(\mathbb{R})$, $t \in \mathbb{R} \setminus \{0\}$ and set $f_t := e^{-it\overline{H_0}} f_0$. We have:*

1. *If $f_0 \in L^1(\mathbb{R})$ then $f_t \in L^\infty(\mathbb{R})$ and*

$$
\|f_t\|_{\infty,\mathbb{R}} \leq \frac{1}{\sqrt{|4\pi t|}} \|f_0\|_{1,\mathbb{R}}.
$$

2. *If $f_0 \in L^1(\mathbb{R})$ then for every bounded (Borel measurable) $S \subseteq \mathbb{R}$*

$$
\|\chi_S f_t\|_{2,\mathbb{R}} \to 0 \quad \text{for} \quad t \to \pm\infty.
$$

3. *$\langle \phi, f_t \rangle \xrightarrow[t\to\pm\infty]{} 0$ for each $\phi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$.*

Proof. 1. We apply Theorem 3.2.1 and obtain

$$
\|f_t\|_{\infty,\mathbb{R}} \leq \frac{1}{\sqrt{|4\pi t|}} \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \left| e^{i\frac{(x-y)^2}{4t}} f_0(y) \right| d\lambda(y) = \frac{1}{\sqrt{|4\pi t|}} \|f_0\|_{1,\mathbb{R}}.
$$

2. We can use 1. to obtain $\|\chi_S f_t\|_{2,\mathbb{R}} \leq \|f_t\|_{\infty,\mathbb{R}} \|\chi_S\|_{2,\mathbb{R}} \to 0$, where we used $\chi_S \in L^2(\mathbb{R})$, since S is bounded.

3. First suppose that $f_0 \in \mathscr{S}(\mathbb{R})$. Then the claim follows from 1.:

$$
|\langle \phi, f_t \rangle| \leq \int_{\mathbb{R}} |\phi||f_t|\, d\lambda \leq \|f_t\|_{\infty,\mathbb{R}} \|\phi\|_{1,\mathbb{R}} \to 0
$$

For arbitrary $f_0 \in L^2(\mathbb{R})$ let $\epsilon > 0$, choose $\tilde{f}_0 \in \mathscr{S}(\mathbb{R})$ such that $\|\phi\|_{1,\mathbb{R}} \|f_0 - \tilde{f}_0\|_{2,\mathbb{R}} < \epsilon$ and $T \in \mathbb{R}$ such that $|\langle \phi, e^{-it\overline{H_0}} \tilde{f}_0 \rangle| < \epsilon$ for all $t \geq T$. Then $|\langle \phi, f_t \rangle| \leq |\langle \phi, e^{-it\overline{H_0}}(f_0 - \tilde{f}_0) \rangle| + |\langle \phi, e^{-it\overline{H_0}} \tilde{f}_0 \rangle| \leq \|\phi\|_{1,\mathbb{R}} \|f_0 - \tilde{f}_0\|_{2,\mathbb{R}} + \epsilon \leq 2\epsilon$ for all $t \geq T$.

□

Remark 3.2.3. The mapping $x \to |f_t(x)|^2$ is commonly interpreted as the probability density of the position of the particle at time t. If we start at time $t = 0$ with a particle that is described by a wave packet $f_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the probability to find this particle in any bounded region Ω will decrease over time. This phenomenon is often called spreading of wave packets.

The next step will be to determine the wave operator $\Omega^l_\pm$ for $l = 1$. We can use Corollary 2.2.7 to get

$$e^{it\overline{H_1}}e^{-it\overline{H_0}} = \left(U_1 e^{it\overline{H_0}}U_1^* + e^{-it}\mathrm{Proj}_{\ker(\overline{a_1})}\right)e^{-it\overline{H_0}}. \tag{3.2}$$

Recall, that, thanks to Lemma 2.2.3, we already know what U_1^* is. Inserting equation 2.13 into equation 3.2 leads to

$$e^{it\overline{H_1}}e^{-it\overline{H_0}} = U_1\frac{\overline{P} - ie^{it\overline{H_0}}M_{th}e^{-it\overline{H_0}}}{(\overline{P}^2+1)^{\frac{1}{2}}} + e^{-it}\mathrm{Proj}_{\ker(\overline{a_1})}e^{-it\overline{H_0}}. \tag{3.3}$$

When taking the SOT-limit $t \to \pm\infty$ the second summand on the right hand side vanishes thanks to Lemma 3.2.2. In order to get control over the first summand we need to take a closer look at the term $e^{it\overline{H_0}}M_{th}e^{-it\overline{H_0}}$. We can start with some intuition from physics. Consider a right-moving wave ϕ and its time evolution $\phi_t := e^{-it\overline{H_0}}\phi$. As ϕ propagates to the right, not many parts of ϕ_t will be located to the left from its starting point and ϕ_t moves to $+\infty$. Since $\tanh(x) \approx 1$ for large x, we have $M_{th}\phi_t \approx \phi_t$ so that $e^{it\overline{H_0}}M_{th}e^{-it\overline{H_0}}\phi \approx \phi$ for large times t and right-moving waves ϕ. Conversely, if ϕ is purely left-moving then $e^{it\overline{H_0}}M_{th}e^{-it\overline{H_0}}\phi \approx -\phi$ since $\tanh(x) \approx -1$ for large negative x. In summary, we would expect that $e^{it\overline{H_0}}M_{th}e^{-it\overline{H_0}}\phi \to E^{\overline{P}}_{\mathbb{R}_+}\phi - E^{\overline{P}}_{\mathbb{R}_-}\phi$ for $t \to +\infty$ (for $t \to -\infty$ the roles of left and right-moving waves are reversed). The following results make this idea precise.

Lemma 3.2.4. *Let $f \in L^\infty(\mathbb{R},\mathbb{R})$ be a bounded real-valued function such that the limit*

$$L_f := \operatorname*{s-lim}_{t\to\pm\infty} e^{it\overline{H_0}}M_f e^{-it\overline{H_0}}$$

exists and $\tanh - f \in L^2(\mathbb{R})$ *(M_f denotes the multiplication operator by f). Then the limit*

$$L_{th} := \operatorname*{s-lim}_{t\to\pm\infty} e^{it\overline{H_0}}M_{th} e^{-it\overline{H_0}}$$

also exists and $L_f = L_{th}$.

Proof. Assume that L_f exists, set $g := \tanh - f$ and suppose $g \in L^2(\mathbb{R})$. We claim that

$$L_g := \operatorname*{s-lim}_{t\to\pm\infty} S_t, \quad S_t := e^{it\overline{H_0}}M_g e^{-it\overline{H_0}}$$

exists and $L_g = 0$. Indeed, for $\phi \in \mathscr{S}(\mathbb{R})$ and $t \in \mathbb{R}$ we have (Lemma 3.2.2)

$$\|S_t\phi\|_{2,\mathbb{R}} = \|M_g e^{-it\overline{H_0}}\phi\|_{2,\mathbb{R}} \le \|g\|_{2,\mathbb{R}}\|e^{-it\overline{H_0}}\phi\|_{\infty,\mathbb{R}} \to 0 \quad \text{for } t \to \pm\infty.$$

Therefore $S_t\phi$ converges. As $\|S_t\| \le \|g\|_{\infty,\mathbb{R}}$ for every $t \in \mathbb{R}$ and $\mathscr{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the Banach-Steinhaus theorem (Appendix A.2.2) assures the existence of L_g and, by the above calculations, we have $L_g = 0$. Because of

$$\begin{aligned} e^{it\overline{H_0}}M_{th}e^{-it\overline{H_0}} &= e^{it\overline{H_0}}M_g e^{-it\overline{H_0}} + e^{it\overline{H_0}}M_f e^{-it\overline{H_0}} \\ &\xrightarrow[t\to\pm\infty]{SOT} L_g + L_f \end{aligned}$$

L_{th} exists and $L_{th} = L_g + L_f = L_f$. ☐

Lemma 3.2.4 allows us to replace tanh in equation 3.3 with any other function $f \in L^\infty(\mathbb{R}, \mathbb{R})$ that looks like tanh (in the sense that $\tanh - f \in L^2(\mathbb{R})$). In particular, we are interested in the choice $f = \chi_{\mathbb{R}_+} - \chi_{\mathbb{R}_-}$, since this function is much easier to handle than tanh. The next step will be to get more insights into the operator $e^{it\overline{H_0}} X e^{-it\overline{H_0}}$ (recall the definition of the position operator X from Remark 1.1.10).

Lemma 3.2.5. *Let $t \in \mathbb{R}$. The operator*

$$X_t := e^{it\overline{H_0}} X e^{-it\overline{H_0}}$$

is well-defined on $D(X_t) := \mathscr{S}(\mathbb{R})$. The operator $W_t := X + 2tP$ on its domain $D(W_t) := D(X) \cap D(P) = \mathscr{S}(\mathbb{R})$ satisfies $X_t = W_t$ for all $t \in \mathbb{R}$. W_t is essentially self-adjoint and $f(\overline{W_t}) = e^{it\overline{H_0}} M_f e^{-it\overline{H_0}}$ for every Borel measurable function $f : \mathbb{R} \to \mathbb{C}$ (as always M_f denotes the multiplication operator with f).

Proof. To see that X_t is well-defined on $\mathscr{S}(\mathbb{R})$ we have to make sure that $e^{it\overline{H_0}} \mathscr{S}(\mathbb{R}) \subseteq D(X) = \mathscr{S}(\mathbb{R})$: Let $f = e^{it\overline{H_0}}\phi$ for some $\phi \in \mathscr{S}(\mathbb{R})$ and set $e_t(p) := e^{itp^2}$. Then

$$f = \mathcal{F}^{-1}(\mathcal{F}f) = \mathcal{F}^{-1}(e_t \mathcal{F}(\phi))$$

and since $e_t \mathcal{F}(\phi) \in \mathscr{S}(\mathbb{R})$ we find $f \in \mathscr{S}(\mathbb{R})$. A direct calculation proves $W_t = X_t$: For $f \in \mathscr{S}(\mathbb{R})$ we have

$$\begin{aligned}
\mathcal{F}(X_t f)(p) &= e_t(p)\mathcal{F}\left(X e^{-it\overline{H_0}} f\right)(p) \\
&= -e_t(p)(-i)\frac{d}{dp}\left(\mathcal{F}\left(e^{-it\overline{H_0}} f\right)\right)(p) \\
&= ie_t(p)\frac{d}{dp}\left(e_{-t}\mathcal{F}(f)\right)(p) \\
&= ie_t(p)\left(e_{-t}(p)\mathcal{F}(f)'(p) - 2itpe_{-t}(p)\mathcal{F}(f)(p)\right) \\
&= i\mathcal{F}(f)'(p) + 2tp\mathcal{F}(f)(p) \\
&= \mathcal{F}(Xf)(p) + 2t\mathcal{F}(Pf)(p)
\end{aligned}$$

and therefore $X_t f = (X + 2tP)f$ for all $f \in \mathscr{S}(\mathbb{R})$. Since X is essentially self-adjoint and $e^{it\overline{H_0}}$ is unitary, Theorem 1.1.8 implies that W_t is essentially self-adjoint with self-adjoint extension $\overline{W_t} = e^{it\overline{H_0}} \overline{X} e^{-it\overline{H_0}}$. Since $f(\overline{X}) = M_f$ for Borel measurable functions f, the relation $f(\overline{W_t}) = e^{it\overline{H_0}} M_f e^{-it\overline{H_0}}$ is also a consequence of Theorem 1.1.8. □

Remark 3.2.6. The factor 2 in the operator $W_t = X + 2tP$ seems a bit "strange", as we would expect to have the relation "Position at time t = Position at $t = 0 + t \cdot$ velocity". However, the additional factor 2 originates from our slightly non-physical definition of the Hamiltonian. In physics the free Hamiltonian (in one dimension) is usually defined as $H_0 = -\frac{1}{2m}\frac{d^2}{dx^2}$, where m describes the mass. In our case, we always assumed $m = 1/2$ so that we only have $H_0 = -\frac{d^2}{dx^2}$. Since "velocity = momentum/m", we then have "velocity = $2 \cdot$ momentum".

As described above, we want to choose $f = \chi_{\mathbb{R}_+} - \chi_{\mathbb{R}_-}$. By the last lemma we then have $e^{it\overline{H_0}} M_f e^{-it\overline{H_0}} = \chi_{\mathbb{R}_+}(\overline{W_t}) - \chi_{\mathbb{R}_-}(\overline{W_t})$. Therefore, we need to determine the projection valued measure of $\overline{W_t}$:

Lemma 3.2.7. *Let $t \in \mathbb{R} \setminus \{0\}$. The spectral measures $E^{\overline{W_t}}$ of the closure $\overline{W_t}$ are given by ($S \in \mathcal{B}(\mathbb{R})$)*

$$E_S^{\overline{W_t}} = M_{e_{-t}} E_S^{2t\overline{P}} M_{e_t},$$

where M_{e_t} denotes the multiplication operator with the function $e_t(p) := e^{i\frac{p^2}{4t}}$ and $E_S^{2t\overline{P}} := \mathcal{F}^{-1} M_{\chi_{S/(2t)}} \mathcal{F}$ is the spectral measure of the operator $2t\overline{P}$ where $S/(2t)$ denotes the set $\{x \in \mathbb{R} : 2tx \in S\}$.

Proof. Clearly $E_S^{2t\overline{P}} = E_{S/(2t)}^{\overline{P}}$ is the projection valued measure of $2t\overline{P}$. Let $E : \mathcal{B}(\mathbb{R}) \to B(L^2(\mathbb{R}))$ be given by $E_S := M_{e_{-t}} E_S^{2t\overline{P}} M_{e_t}$ for $S \in \mathcal{B}(\mathbb{R})$. Then E is a projection valued measure because M_{e_t} is unitary and $E^{2t\overline{P}}$ is a projection valued measure. Therefore, the operator

$$A := \int_{\mathbb{R}} \lambda \, dE_\lambda$$

is self-adjoint and satisfies $A = M_{e_{-t}}(2t\overline{P})M_{e_t}$ (according to Scholium 1.1.9). Hence, $D(A) = M_{e_{-t}} D(\overline{P})$ and in particular $\mathscr{S}(\mathbb{R}) \subseteq D(A)$. Therefore, for $f \in \mathscr{S}(\mathbb{R})$ we get

$$\begin{aligned}
(Af)(x) &= 2te_{-t}(x)\overline{P}(e_t f)(x) \\
&= -2tie_{-t}\left(e_t'(x)f(x) + e_t(x)f'(x)\right) \\
&= -2ti\left(\frac{2ix}{4t}f(x) + f'(x)\right) \\
&= xf(x) - 2tif'(x) = (X + 2tP)\, f(x) = W_t f(x).
\end{aligned}$$

Hence, $W_t \subset A$ and by a standard argument we did many times before (W_t is essentially self-adjoint therefore the self-adjoint extension is unique) we find that $\overline{W_t} = A$ and $E^{\overline{W_t}} = E$. □

Corollary 3.2.8. *The spectral measure $E^{\overline{W_t}}(S)$ for $S \in \{\mathbb{R}_+, \mathbb{R}_-\}$ has the following asymptotics*

$$\begin{aligned}
\operatorname*{s-lim}_{t\to\infty} E^{\overline{W_t}}(\mathbb{R}_\pm) &= E^{\overline{P}}(\mathbb{R}_\pm) \\
\operatorname*{s-lim}_{t\to-\infty} E^{\overline{W_t}}(\mathbb{R}_\pm) &= E^{\overline{P}}(\mathbb{R}_\mp),
\end{aligned}$$

where $E^{\overline{P}}(S)$ denotes the spectral measure of the momentum operator $\overline{P}$.

Proof. For $t \in \mathbb{R} \setminus \{0\}$ we have

$$\frac{\mathbb{R}_\pm}{2t} = \begin{cases} \mathbb{R}_\pm & \text{if } t > 0 \\ \mathbb{R}_\mp & \text{if } t < 0 \end{cases}$$

and hence

$$E^{\overline{W_t}}(\mathbb{R}_\pm) = M_{e_{-t}} \mathcal{F}^{-1} M_{\chi_{\mathbb{R}_\pm/(2t)}} \mathcal{F} M_{e_t} = \begin{cases} M_{e_{-t}} E^{\overline{P}}(\mathbb{R}_\pm) M_{e_t} & \text{if } t > 0 \\ M_{e_{-t}} E^{\overline{P}}(\mathbb{R}_\mp) M_{e_t} & \text{if } t < 0 \end{cases}.$$

Next we claim that $M_{e_{\pm t}} \to 1$ in SOT. To prove this, consider $f \in L^2(\mathbb{R})$. As $e_{\pm t} \to 1$ pointwise we have

$$\|(M_{e_{\pm t}} - 1)f\|_{2,\mathbb{R}}^2 = \int_{\mathbb{R}} |e_{\pm t}(x) - 1|^2 |f(x)|^2 \, dx \xrightarrow[t \to \pm\infty]{} 0$$

thanks to Lebesgue Dominated convergence (one can choose $4|f|^2$ as a majorant). With this we can prove the above-mentioned asymptotics of $E^{\overline{W_t}}$. Again, let $f \in L^2(\mathbb{R})$. We find for $t > 0$:

$$\begin{aligned}
&\|E^{\overline{W_t}}(\mathbb{R}_\pm)f - E^{\overline{P}}(\mathbb{R}_\pm)f\|_{2,\mathbb{R}} = \|M_{e_{-t}}E^{\overline{P}}(\mathbb{R}_\pm)M_{e_t}f - E^{\overline{P}}(\mathbb{R}_\pm)f\|_{2,\mathbb{R}} \\
&\le \|M_{e_{-t}}E^{\overline{P}}(\mathbb{R}_\pm)M_{e_t}f - M_{e_{-t}}E^{\overline{P}}(\mathbb{R}_\pm)f\|_{2,\mathbb{R}} + \|M_{e_{-t}}E^{\overline{P}}(\mathbb{R}_\pm)f - E^{\overline{P}}(\mathbb{R}_\pm)f\|_{2,\mathbb{R}} \\
&\le \|(M_{e_t} - 1)\, f\|_{2,\mathbb{R}} + \|\left(M_{e_{-t}} - 1\right) E^{\overline{P}}(\mathbb{R}_\pm)f\|_{2,\mathbb{R}} \\
&\to 0 \quad \text{for } t \to -\infty
\end{aligned}$$

And likewise for negative times t:

$$\begin{aligned}
&\|E^{\overline{W_t}}(\mathbb{R}_\pm)f - E^{\overline{P}}(\mathbb{R}_\mp)f\|_{2,\mathbb{R}} = \|M_{e_{-t}}E^{\overline{P}}(\mathbb{R}_\mp)M_{e_t}f - E^{\overline{P}}(\mathbb{R}_\mp)f\|_{2,\mathbb{R}} \\
&\le \|(M_{e_t} - 1)\, f\|_{2,\mathbb{R}} + \|\left(M_{e_{-t}} - 1\right) E^{\overline{P}}(\mathbb{R}_\mp)f\|_{2,\mathbb{R}} \\
&\to 0 \quad \text{for } t \to \infty
\end{aligned}$$

□

Now we have everything together to give our physical intuition from the beginning of this section a mathematically precise foundation.

Theorem 3.2.9. *For $l = 1$ the wave operators $\Omega_\pm^l$ exist and are given by*

$$\Omega_\pm^1 = U_1 \frac{\overline{P} - i\left(E_{\mathbb{R}_\pm}^{\overline{P}} - E_{\mathbb{R}_\mp}^{\overline{P}}\right)}{\left(\overline{P}^2 + 1\right)^{\frac{1}{2}}}.$$

Proof. From Corollary 2.2.7 we derive

$$e^{it\overline{H_1}}e^{-it\overline{H_0}} = U_1 e^{it\overline{H_0}} U_1^* e^{-it\overline{H_0}} + e^{-it}\mathrm{Proj}_{\ker(\overline{a_1})} e^{-it\overline{H_0}}.$$

Lemma 2.2.3 (equation 2.13) allows us to simplify the above equation to

$$\begin{aligned}
e^{it\overline{H_1}}e^{-it\overline{H_0}} &= U_1 e^{it\overline{H_0}} \left(\frac{\overline{P}}{(\overline{P}^2 + 1)^{\frac{1}{2}}} - i(\overline{P}^2 + 1)^{-\frac{1}{2}} M_{th} \right) e^{-it\overline{H_0}} + C_t \\
&= U_1 \left(\frac{\overline{P}}{(\overline{P}^2 + 1)^{\frac{1}{2}}} - i(\overline{P}^2 + 1)^{-\frac{1}{2}} e^{it\overline{H_0}} M_{th} e^{-it\overline{H_0}} \right) + C_t,
\end{aligned}$$

where we set $C_t \coloneqq e^{-it}\mathrm{Proj}_{\ker(\overline{a_1})} e^{-it\overline{H_0}}$. Our goal is to use Lemma 3.2.4 to replace the multiplication with tanh by the multiplication with $g \coloneqq \chi_{(0,\infty)} - \chi_{(-\infty,0)}$. To this end, observe that $g \in L^\infty(\mathbb{R})$ and $\tanh - g \in L^2(\mathbb{R})$, which can be seen from the estimate

$$\begin{aligned}
|\tanh(x) - g(x)| &= \left|(\tanh(x) - 1)\, \chi_{(0,\infty)}(x) + (\tanh(x) + 1)\, \chi_{(-\infty,0)}(x)\right| \\
&= \left| -\frac{2}{e^{2x} + 1} \chi_{(0,\infty)}(x) + \frac{2}{e^{-2x} + 1} \chi_{(-\infty,0)}(x) \right| \\
&\le 2\chi_{(0,\infty)}(x) e^{-2x} + 2\chi_{(-\infty,0)}(x) e^{2x},
\end{aligned}$$

where we have used that $\tanh(x) = 1 - \frac{2}{e^{2x}+1}$. Thanks to Lemma 3.2.5 we have

$$e^{it\overline{H_0}} M_g e^{-it\overline{H_0}} = g(\overline{W_t}) = E^{\overline{W_t}}(\mathbb{R}_+) - E^{\overline{W_t}}(\mathbb{R}_-).$$

Applying Corollary 3.2.8 leads to

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\overline{H_0}} M_g e^{-it\overline{H_0}} = E^{\overline{P}}(\mathbb{R}_\pm) - E^{\overline{P}}(\mathbb{R}_\mp).$$

Back to Lemma 3.2.4, we therefore know that

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\overline{H_0}} M_{th} e^{-it\overline{H_0}} = E^{\overline{P}}(\mathbb{R}_\pm) - E^{\overline{P}}(\mathbb{R}_\mp).$$

Next we have to determine $\operatorname{s-lim}_{t\to\pm\infty} C_t$. Let $\phi \in \ker(\overline{a_l})$ be a normalized eigenvector. Then for $f \in L^2(\mathbb{R})$ we have

$$\operatorname{Proj}_{\ker(\overline{a_l})} f = \langle \phi, f\rangle \phi,$$

since $\ker(\overline{a_l})$ is one-dimensional (Corollary 2.1.6). Furthermore, thanks to Corollary 2.1.6 $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and therefore, Lemma 3.2.2 implies

$$C_t f = e^{-it}\langle \phi, e^{it\overline{H_0}} f\rangle \phi \xrightarrow[t\to\pm\infty]{} 0 \implies \operatorname*{s-lim}_{t\to\pm\infty} C_t = 0. \tag{3.4}$$

Putting everything together, we arrive at

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\overline{H_1}} e^{-it\overline{H_0}} = U_1 \left(\frac{\overline{P}}{(\overline{P}^2+1)^{\frac{1}{2}}} - i(\overline{P}^2+1)^{-\frac{1}{2}} \left(E^{\overline{P}}(\mathbb{R}_\pm) - E^{\overline{P}}(\mathbb{R}_\mp) \right) \right).$$

□

Corollary 3.2.10. *For $l = 1$ the scattering operator S_l is given by*

$$S_1 = \frac{\overline{P} - i}{\overline{P} + i} E^{\overline{P}}(\mathbb{R}_-) + \frac{\overline{P} + i}{\overline{P} - i} E^{\overline{P}}(\mathbb{R}_+).$$

Proof. By Theorem 3.2.9 (and using that $U_1^* U_1 = 1$ according to Lemma 2.2.3) we obtain

$$\begin{aligned}
S_1 = \left(\Omega_+^1\right)^* \Omega_-^1 &= \left(\frac{\overline{P} - i(E^{\overline{P}}(\mathbb{R}_-) - E^{\overline{P}}(\mathbb{R}_+))}{\left(\overline{P}^2+1\right)^{\frac{1}{2}}} \right)^2 \\
&= \frac{\overline{P}^2 - 2i\overline{P}\left(E^{\overline{P}}(\mathbb{R}_-) - E^{\overline{P}}(\mathbb{R}_+)\right) - \left(E^{\overline{P}}(\mathbb{R}_-) - E^{\overline{P}}(\mathbb{R}_+)\right)^2}{\overline{P}^2+1} \\
&= \frac{\overline{P}^2 - 2i\overline{P}\left(E^{\overline{P}}(\mathbb{R}_-) - E^{\overline{P}}(\mathbb{R}_+)\right) - 1}{\overline{P}^2+1} \\
&= \frac{\overline{P}^2\left(E^{\overline{P}}(\mathbb{R}_-) + E^{\overline{P}}(\mathbb{R}_+)\right) - 2i\overline{P}\left(E^{\overline{P}}(\mathbb{R}_-) - E^{\overline{P}}(\mathbb{R}_+)\right) - \left(E^{\overline{P}}(\mathbb{R}_-) + E^{\overline{P}}(\mathbb{R}_+)\right)}{\overline{P}^2+1} \\
&= \frac{\left(\overline{P}-i\right)^2 E^{\overline{P}}(\mathbb{R}_-) + \left(\overline{P}+i\right)^2 E^{\overline{P}}(\mathbb{R}_+)}{\overline{P}^2+1}.
\end{aligned}$$

The relation $\overline{P}^2 + 1 = \left(\overline{P}+i\right)\left(\overline{P}-i\right)$ finishes the proof. □

In the next steps we want to generalize these results to arbitrary $l \in \mathbb{N}$. We are going to use the same ideas recursively and replace every occurrence of $\overline{H_l}$ with $\overline{H_{l-1}}$ with the help of Lemma 2.2.4, until everything is expressible in terms of $\overline{H_0}$ and $\overline{P}$. However, we have to do a bit of bookkeeping to keep track of all the operators U_l, which will definitely show up during this process. Furthermore, we are going to make use Lemma 2.2.3. However, in contrast to the proof of Theorem 3.2.9, we no longer can use equation 2.13. We have to deal with the more general equation 2.12. Therefore, we must take care of the restrictions to $D(\overline{a_l})$. To overcome this issue, we first look at "wave operators" which might not be defined everywhere and continue from there.

Definition 3.2.1. Let $l \in \mathbb{N} \cup \{0\}$. Set

$$D(\tilde{\Omega}^l_\pm) := \{f \in L^2(\mathbb{R}) \,:\, \lim_{t\to\pm\infty} e^{it\overline{H_l}} e^{-it\overline{H_0}} f \text{ exists}\}$$

and define the linear operators

$$\tilde{\Omega}^l_\pm : D(\tilde{\Omega}^l_\pm) \to L^2(\mathbb{R}), \quad \tilde{\Omega}^l_\pm := \lim_{t\to\pm\infty} e^{it\overline{H_l}} e^{-it\overline{H_0}} f.$$

Remark 3.2.11. If $D(\tilde{\Omega}^l_\pm) \subseteq L^2(\mathbb{R})$ is a dense subspace and $\tilde{\Omega}^l_\pm$ is bounded (on that subspace), then it can be uniquely extended to a bounded operator on $L^2(\mathbb{R})$. In that case, $\Omega^l_\pm$ exists and is equal to the extension of $\tilde{\Omega}^l_\pm$.

The first question is, to what extent $\tilde{\Omega}^l_\pm$ is determined by $\Omega^{l-1}_\pm$ (in case the latter one exists).

Lemma 3.2.12. *Let $l \in \mathbb{N}$ and suppose that $\Omega^{l-1}_\pm$ exists. Then for $f \in \mathscr{S}(\mathbb{R})$ the limit $\lim_{t\to\pm\infty} e^{it\overline{H_l}} e^{-it\overline{H_0}} f$ exists, so that $\mathscr{S}(\mathbb{R}) \subseteq D(\tilde{\Omega}^l_\pm)$. Furthermore, on $\mathscr{S}(\mathbb{R})$, $\tilde{\Omega}^l_\pm$ is given by*

$$\tilde{\Omega}^l_\pm\big|_{\mathscr{S}(\mathbb{R})} = U_l \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} \Omega^{l-1}_\pm \left(P - il\left(E^{\overline{P}}_{\mathbb{R}_\pm} - E^{\overline{P}}_{\mathbb{R}_\mp}\right)\right).$$

Proof. By Lemma 2.2.4 we have

$$e^{it\overline{H_l}} e^{-it\overline{H_0}} = \left(U_l e^{it\overline{H_{l-1}}} U_l^* + e^{-itl^2} \mathrm{Proj}_{\ker(\overline{a_l})}\right) e^{-it\overline{H_0}} \tag{3.5}$$

$$= U_l e^{it\overline{H_{l-1}}} U_l^* e^{-it\overline{H_0}} + C_{t,l}, \tag{3.6}$$

where $C_{t,l} := e^{-itl^2} \mathrm{Proj}_{\ker(\overline{a_l})} e^{-it\overline{H_0}}$. We can reuse the ideas in the proof of Theorem 3.2.9 twice: First, we notice that equation 3.4 implies $\text{s-lim}_{t\to\pm\infty} C_{t,l} = 0$. Second, equation 2.12 leads to

$$\begin{aligned}
&U_l e^{it\overline{H_{l-1}}} U_l^* e^{-it\overline{H_0}} f \\
&= U_l \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} e^{it\overline{H_{l-1}}} \left(\overline{P} - ilM_{th}\right) e^{-it\overline{H_0}} f \\
&= U_l \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} \left(e^{it\overline{H_{l-1}}} e^{-it\overline{H_0}} P - il e^{it\overline{H_{l-1}}} e^{-it\overline{H_0}} e^{it\overline{H_0}} M_{th} e^{-it\overline{H_0}}\right) f
\end{aligned}$$

for $f \in \mathscr{S}(\mathbb{R})$. Observe that (by assumption) the limit $\text{s-lim}_{t\to\pm\infty} e^{it\overline{H_{l-1}}} e^{-it\overline{H_0}} = \Omega^{l-1}_\pm$ exists so that we can use the proof of Theorem 3.2.9 (equation 3.2) one last time to conclude

$$U_l e^{it\overline{H_{l-1}}} U_l^* e^{-it\overline{H_0}} f \xrightarrow[t\to\pm\infty]{} U_l \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} \Omega^{l-1}_\pm \left(P - il\left(E^{\overline{P}}_{\mathbb{R}_\pm} - E^{\overline{P}}_{\mathbb{R}_\mp}\right)\right) f.$$

Putting everything together, equation 3.6 applied to $f \in \mathscr{S}(\mathbb{R})$ evaluates in the limit $t \to \pm\infty$ to

$$e^{it\overline{H_l}} e^{-it\overline{H_0}} f \xrightarrow[t\to\pm\infty]{} U_l \left(\overline{H_{l-1}} + l^2\right)^{-\frac{1}{2}} \Omega_\pm^{l-1} \left(P - il \left(E^{\overline{P}}_{\mathbb{R}_\pm} - E^{\overline{P}}_{\mathbb{R}_\mp}\right)\right) f,$$

which completes the proof. □

As we already have knowledge about $\Omega_\pm^l$ for $l = 1$, we can use induction on l to get that $\tilde{\Omega}_\pm^{l+1}$ is densely defined. We can then try to use Remark 3.2.11 to extend $\tilde{\Omega}_\pm^{l+1}$ boundedly to $\Omega_\pm^l$. Before doing so, we need a bit more knowledge about $\Omega_\pm^l$.

Lemma 3.2.13. *Let $l \in \mathbb{N}$ and suppose that $\Omega_\pm^l$ exists. Then* $\operatorname{Ran}(\Omega_\pm^l) \subseteq \ker(\overline{a_l})^\perp$.

Proof. Let $g \in L^2(\mathbb{R})$, $f := \Omega_\pm^l g \in \operatorname{Ran}(\Omega_\pm^l)$ and $\psi \in \ker(\overline{a_l})$. Then ψ is an eigenvector of $\overline{H_l}$ with eigenvalue $-l^2$ (Corollary 2.1.4). Therefore,

$$\begin{aligned}\langle \psi, f\rangle &= \lim_{t\to\pm\infty} \langle e^{it\overline{H_0}} e^{-it\overline{H_l}} \psi, g\rangle \\ &= \lim_{t\to\pm\infty} e^{itl^2} \langle \psi, e^{-it\overline{H_0}} g\rangle = 0,\end{aligned}$$

where the last equality is justified by Lemma 3.2.2 (again, $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by Corollary 2.1.6). □

Theorem 3.2.14. *For each $l \in \mathbb{N}$ the wave operators $\Omega_\pm^l$ exist and are given by*

$$\Omega_\pm^l = \left(\prod_{j=0}^{l-1} U_{l-j}\right) \left(\prod_{j=1}^{l} \frac{\overline{P} - ij\left(E^{\overline{P}}_{\mathbb{R}_\pm} - E^{\overline{P}}_{\mathbb{R}_\mp}\right)}{\left(\overline{P}^2 + j^2\right)^{\frac{1}{2}}}\right). \tag{3.7}$$

Proof. We prove the claim by induction on l. For $l = 1$ Theorem 3.2.9 proves the claim. Let $l \in \mathbb{N}$ be such that for each $j \in \{1, ..., l\}$ $\Omega_\pm^j$ exists and is given by equation 3.7. Then, by Lemma 3.2.12, $\mathscr{S}(\mathbb{R}) \subseteq D(\tilde{\Omega}_\pm^{l+1})$ and

$$\tilde{\Omega}_\pm^{l+1}\Big|_{\mathscr{S}(\mathbb{R})} = U_{l+1} \left(\overline{H_l} + (l+1)^2\right)^{-\frac{1}{2}} \Omega_\pm^l \left(P - i(l+1) \left(E^{\overline{P}}_{\mathbb{R}_\pm} - E^{\overline{P}}_{\mathbb{R}_\mp}\right)\right). \tag{3.8}$$

Using the induction hypothesis and applying Lemma 2.2.4 on $\left(\overline{H_j} + (l+1)^2\right)^{-\frac{1}{2}}$ yields

$$\begin{aligned}&\left(\overline{H_j} + (l+1)^2\right)^{-\frac{1}{2}} \Omega_\pm^j \\ &= U_j \left(\overline{H_{j-1}} + (l+1)^2\right)^{-\frac{1}{2}} U_j^* U_j \Omega_\pm^{j-1} \frac{\overline{P} - ij(E^{\overline{P}}_\pm - E^{\overline{P}}_\mp)}{(\overline{P}^2 + j^2)^{\frac{1}{2}}} \\ &+ (-j^2 + (l+1)^2)^{-\frac{1}{2}} \operatorname{Proj}_{\ker(\overline{a_j})} \Omega_\pm^j.\end{aligned}$$

According to Lemma 3.2.13 and the fact that U_j is an isometry, the last equation simplifies to

$$\left(\overline{H_j} + (l+1)^2\right)^{-\frac{1}{2}} \Omega_\pm^j = U_j \left(\overline{H_{j-1}} + (l+1)^2\right)^{-\frac{1}{2}} \Omega_\pm^{j-1} \frac{\overline{P} - ij(E^{\overline{P}}_{\mathbb{R}_\pm} - E^{\overline{P}}_{\mathbb{R}_\mp})}{(\overline{P}^2 + j^2)^{\frac{1}{2}}}.$$

Recursively, we obtain (recall that $\Omega_\pm^0 = 1$)

$$\left(\overline{H_l} + (l+1)^2\right)^{-\frac{1}{2}} \Omega_\pm^l = \left(\prod_{j=0}^{l-1} U_{l-j}\right) \left(\overline{H_0} + (l+1)^2\right)^{-\frac{1}{2}} \left(\prod_{j=1}^{l} \frac{\overline{P} - ij(E_{\mathbb{R}_\pm}^{\overline{P}} - E_{\mathbb{R}_\mp}^{\overline{P}})}{(\overline{P}^2 + j^2)^{\frac{1}{2}}}\right)$$
$$= \left(\prod_{j=0}^{l-1} U_{l-j}\right) \left(\prod_{j=1}^{l} \frac{\overline{P} - ij(E_{\mathbb{R}_\pm}^{\overline{P}} - E_{\mathbb{R}_\mp}^{\overline{P}})}{(\overline{P}^2 + j^2)^{\frac{1}{2}}}\right) \left(\overline{H_0} + (l+1)^2\right)^{-\frac{1}{2}},$$

where the last equality is justified by the fact that the interchanged operators are all of the form $g(\overline{P})$ for bounded Borel functions g. Substituting this into equation 3.8 leads to

$$\tilde{\Omega}_\pm^{l+1}\big|_{\mathscr{S}(\mathbb{R})} = \left(\prod_{j=0}^{l} U_{l+1-j}\right) \left(\prod_{j=1}^{l+1} \frac{\overline{P} - ij(E_{\mathbb{R}_\pm}^{\overline{P}} - E_{\mathbb{R}_\mp}^{\overline{P}})}{(\overline{P}^2 + j^2)^{\frac{1}{2}}}\right).$$

The right-hand side describes a bounded operator on $L^2(\mathbb{R})$. Therefore, the right-hand side is a bounded extension of $\tilde{\Omega}_\pm^{l+1}$ to all of $L^2(\mathbb{R})$. Thus, thanks to Remark 3.2.11, $\Omega_\pm^{l+1}$ exists and fulfills equation 3.7. □

Remark 3.2.15. Observe that we recursively obtain $\Omega_\pm^l$ via

$$\Omega_\pm^l = U_l \Omega_\pm^{l-1} \frac{\overline{P} - il(E_{\mathbb{R}_\pm}^{\overline{P}} - E_{\mathbb{R}_\mp}^{\overline{P}})}{(\overline{P}^2 + l^2)^{\frac{1}{2}}},$$

as can be directly seen by splitting up the first factor in the first product and the last factor in the second product of equation 3.7.

Again, as in the case $l = 1$, the scattering operator S_l now follows easily.

Theorem 3.2.16. *For $l \in \mathbb{N}$ the scattering operator S_l is given by*

$$S_l = \left(\prod_{j=1}^{l} \frac{\overline{P} - ij}{\overline{P} + ij}\right) E_{\mathbb{R}_-}^{\overline{P}} + \left(\prod_{j=1}^{l} \frac{\overline{P} + ij}{\overline{P} - ij}\right) E_{\mathbb{R}_+}^{\overline{P}}.$$

Moreover, S_l is unitary.

Proof. We use Theorem 3.2.14 to obtain

$$S_l = \left(\Omega_+^l\right)^* \Omega_-^l \tag{3.9}$$
$$= \left(\prod_{j=1}^{l} \frac{\overline{P} - ij\left(E_{\mathbb{R}_+}^{\overline{P}} - E_{\mathbb{R}_-}^{\overline{P}}\right)}{\left(\overline{P}^2 + j^2\right)^{\frac{1}{2}}}\right)^* \Gamma_l \left(\prod_{j=1}^{l} \frac{\overline{P} - ij\left(E_{\mathbb{R}_-}^{\overline{P}} - E_{\mathbb{R}_+}^{\overline{P}}\right)}{\left(\overline{P}^2 + j^2\right)^{\frac{1}{2}}}\right), \tag{3.10}$$

where

$$\Gamma_l \coloneqq \left(\prod_{j=0}^{l-1} U_{l-j}\right)^* \left(\prod_{j=0}^{l-1} U_{l-j}\right) = \left(\prod_{j=1}^{l} U_j^*\right) \left(\prod_{j=0}^{l-1} U_{l-j}\right) = 1,$$

where we used that the operators U_j are isometric (Lemma 2.2.3). Since all factors are of the form $g(\overline{P})$ for bounded Borel functions g, they all commute pairwise. This allows us to rewrite equation 3.10 as

$$\begin{aligned}
S_l &= \left(\prod_{j=1}^{l} \frac{\overline{P} + ij\left(E_{\mathbb{R}_+}^{\overline{P}} - E_{\mathbb{R}_-}^{\overline{P}}\right)}{\left(\overline{P}^2 + j^2\right)^{\frac{1}{2}}}\right)\left(\prod_{j=1}^{l} \frac{\overline{P} - ij\left(E_{\mathbb{R}_-}^{\overline{P}} - E_{\mathbb{R}_+}^{\overline{P}}\right)}{\left(\overline{P}^2 + j^2\right)^{\frac{1}{2}}}\right) \\
&= \prod_{j=1}^{l} \frac{\overline{P}^2 - 2\overline{P}ij\left(E_{\mathbb{R}_-}^{\overline{P}} - E_{\mathbb{R}_+}^{\overline{P}}\right) - j^2\left(E_{\mathbb{R}_-}^{\overline{P}} - E_{\mathbb{R}_+}^{\overline{P}}\right)^2}{\overline{P}^2 + j^2} \\
&= \prod_{j=1}^{l} \frac{\overline{P}^2 - 2\overline{P}ij\left(E_{\mathbb{R}_-}^{\overline{P}} - E_{\mathbb{R}_+}^{\overline{P}}\right) - j^2}{\overline{P}^2 + j^2}.
\end{aligned}$$

Again, as in the proof of Corollary 3.2.10, we can use $1 = E_{\mathbb{R}_-}^{\overline{P}} + E_{\mathbb{R}_+}^{\overline{P}}$ to get that

$$\begin{aligned}
S_l &= \prod_{j=1}^{l} \frac{\left(\overline{P} - ij\right)^2 E_{\mathbb{R}_-}^{\overline{P}} + \left(\overline{P} + ij\right)^2 E_{\mathbb{R}_+}^{\overline{P}}}{\left(\overline{P} - ij\right)\left(\overline{P} + ij\right)} \\
&= \prod_{j=1}^{l} \left(\frac{\overline{P} - ij}{\overline{P} + ij} E_{\mathbb{R}_-}^{\overline{P}} + \frac{\overline{P} + ij}{\overline{P} - ij} E_{\mathbb{R}_+}^{\overline{P}}\right).
\end{aligned}$$

Since $E_{\mathbb{R}_\pm}^{\overline{P}} E_{\mathbb{R}_\mp}^{\overline{P}} = 0$. we can simplify this even further:

$$S_l = \left(\prod_{j=1}^{l} \frac{\overline{P} - ij}{\overline{P} + ij}\right) E_{\mathbb{R}_-}^{\overline{P}} + \left(\prod_{j=1}^{l} \frac{\overline{P} + ij}{\overline{P} - ij}\right) E_{\mathbb{R}_+}^{\overline{P}}$$

The unitarity of S_l follows easily:

$$\begin{aligned}
S_l^* S_l &= \left(\prod_{j=1}^{l} \frac{\overline{P} + ij}{\overline{P} - ij}\right)\left(\prod_{j=1}^{l} \frac{\overline{P} - ij}{\overline{P} + ij}\right) E_{\mathbb{R}_-}^{\overline{P}} + \left(\prod_{j=1}^{l} \frac{\overline{P} - ij}{\overline{P} + ij}\right)\left(\prod_{j=1}^{l} \frac{\overline{P} + ij}{\overline{P} - ij}\right) E_{\mathbb{R}_+}^{\overline{P}} \\
&= \left(\prod_{j=1}^{l} \frac{\overline{P} + ij}{\overline{P} - ij}\right)\left(\prod_{j=1}^{l} \frac{\overline{P} - ij}{\overline{P} + ij}\right)\left(E_{\mathbb{R}_-}^{\overline{P}} + E_{\mathbb{R}_+}^{\overline{P}}\right) \\
&= 1.
\end{aligned}$$

The same calculations show $S_l S_l^* = 1$. □

We end this section with a physical remark: Suppose, we have a purely right-moving incoming wave $\psi \in L^2(\mathbb{R})$, i.e. $\psi = \psi_+$ (recall the definition of $\psi_\pm := E_{\mathbb{R}_\pm}^{\overline{P}}\psi$ from the beginning of the chapter). Then the outgoing state $\tilde{\psi} := S_l\psi$ is given by

$$\tilde{\psi} = \left(\prod_{j=1}^{l} \frac{\overline{P} + ij}{\overline{P} - ij}\right)\psi.$$

Therefore,

$$\left(\mathcal{F}\tilde{\psi}\right)(k) = t_l(k)\left(\mathcal{F}\psi\right)(k),$$

where $t_l(k) := \prod_{j=1}^{l} \frac{k+ij}{k-ij}$. Since $\operatorname{supp}(\mathcal{F}\psi) \subseteq \mathbb{R}_+$, we also have $\operatorname{supp}\left(\mathcal{F}\tilde{\psi}\right) \subseteq \mathbb{R}_+$. That means, S maps right-moving waves to right-moving waves. The same line of reasoning shows that S_l maps left-moving to left-moving waves. Hence, the Pöschl-Teller-potential is transparent. The coefficient t_l is called transmission coefficient.

Conclusion and Outlook

In the present work we have taken a look at Schrödinger operators with bounded potentials. In particular, this class of potentials covers the Pöschl-Teller potentials. We were able to use the methods developed in the first and second chapters to calculate the Møller operators and the scattering operator of the Pöschl-Teller potential explicitly. This allowed us to prove that the Pöschl-Teller potentials are transparent. Therefore, we can say that the goals formulated in the preamble have been achieved. Admittedly, bounded potentials do not cover all the physically important potentials by far.

The main results of this thesis are Theorem 2.2.9 and Theorem 3.2.16, which give rise to interesting follow-up questions, such as the following: Can the products in Theorem 2.2.9 be simplified? Based on the explicit form of the Møeller operators, can we say something about completeness? What is the special form of the transmission coefficient all about and is there any reason to expect the eigenvalues to appear? Questions like the third one have already been answered in literature. Questions like the first two, however, have not been studied as far as I know. Due to the limited time and scope of this work, I was unable to answer these questions. However, perhaps they can be examined in a future work.

Appendix

A.1 Prerequisites in Spectral Theory

Lemma A.1.1 ([6], page 256)**.** *Let $T : D(T) \to \mathscr{H}$ be a densely defined symmetric operator. Then the following statements are equivalent:*

1. *T is self-adjoint.*
2. *$ker(T^* \pm i) = \{0\}$ and T is closed.*
3. *$Ran(T \pm i) = \mathscr{H}$.*

Lemma A.1.2 ([6], page 257)**.** *Let $T : D(T) \to \mathscr{H}$ be a densely defined symmetric operator. Then the following statements are equivalent:*

1. *T is essentially self-adjoint.*
2. *$ker(T^* \pm i) = \{0\}$.*
3. *$Ran(T \pm i)$ is dense in $\mathscr{H}$.*

Lemma A.1.3 ([11], page 76)**.** *Let $\lambda \in \mathbb{C}$ and $T : D(T) \to \mathscr{H}$ be a closed densely defined operator. If there exists a seqeuence $(x_j)_{j\in\mathbb{N}} \subseteq D(T)$ such that $\|x_j\| = 1$ for all $j \in \mathbb{N}$ and $(T - \lambda)x \to 0$ (in norm) for $x \to \infty$ then $\lambda \in \sigma(T)$. Such a sequence is called a Weyl sequence.*

A.2 Prerequisites in Functional Analysis

Theorem A.2.1 ([2], page 284)**.** *Let $n \in \mathbb{N}$, $1 \leq p < \infty$ and $m \in \mathbb{N}$. Then, for all $k \in \mathbb{N} \cup \{0\}$ with $k \leq m - \frac{n}{p}$ one has $H^{m,p}(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$.*

Theorem A.2.2 ([13], page 174)**.** *Let X, Y be Banach spaces and $(T_j)_{j\in\mathbb{N}} \subseteq B(X,Y)$ be bounded linear operators from X to Y. There exists $T \in B(X,Y)$ such that $T_j \to T$ in SOT if and only if the following two conditions are fullfilled:*

1. *The sequence $(T_j)_{j\in\mathbb{N}}$ is bounded, i.e. $\sup_{j\in\mathbb{N}}\|T_j\| < \infty$.*
2. *There exists a dense subset $D \subseteq X$ such that $(T_j x)_{j\in\mathbb{N}} \subseteq Y$ converges for every $x \in D$.*

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