KMS States on Crossed Products by Finite Groups

Twisted KMS Functionals and Operator-Valued States

Master's Thesis

for the degree of

Master of Science (M.Sc.)

Mathematics

at the Faculty of Sciences of
Friedrich-Alexander-Universität
Erlangen-Nürnberg

submitted on August 23, 2024

by **Johannes Große**

Supervisor: Prof. Dr. Gandalf Lechner

Friedrich-Alexander-Universität
Naturwissenschaftliche Fakultät

Contents

1 Introduction **3**

2 Crossed Products by Finite Groups **9**

3 KMS States and their Representation Theory **15**

3.1 Introduction to KMS States 15

3.2 Representation Theory of KMS States 18

3.3 Introduction to Modular Theory 20

3.4 The Structure of the Set of KMS States 23

4 KMS States on Crossed Products **27**

4.1 Canonical Extension of a KMS State 27

4.2 GNS Representation of the Crossed Product 31

4.3 Twisted KMS Functionals 35

4.4 Twisted Center of a Von Neumann Algebra 41

4.5 $\mathfrak{M}$-Valued Equivariant γ-Inner States on Finite Groups 48

4.6 The Structure of Equivariant γ-Inner $\mathfrak{M}$-valued States 55

4.7 Asymptotically Abelian Dynamics 59

5 The CAR Algebra **65**

5.1 The CAR Dynamical System 65

5.2 KMS States on the CAR Algebra 69

6 Examples from Mathematical Physics **79**

7 Summary and Outlook **83**

A Appendix **85**

Abstract

The KMS states on an observable algebra model its thermal equilibrium behavior. The crossed product of an algebra of observable by a finite subgroup of its symmetries is an enlarged observable algebra. We combine these two physical notions.

Extensions of a KMS state ω of a unital C^*-algebra $\mathcal{A}$ to the crossed product by a finite group G are characterized. The extensions are parameterized in terms of families of twisted KMS functionals of $\mathcal{A}$ or equivalently by operator-valued states on G whose image lies in the twisted center of the von Neumann algebra generated by the GNS representation corresponding to ω. If the G-action is weakly inner, a decomposition of the extensions of the KMS state is found in terms of the characters of G. If ω is the unique KMS state, the dynamics is ω-weakly asymptotically abelian and the G-action is non-trivial, the canonical extension is the unique extension to the crossed product.

As a particular class of examples, KMS states on crossed products of (self-dual) CAR algebras with dynamics and G-action given by Bogoliubov automorphisms are analyzed in detail. Depending on a Gibbs type condition involving the odd part of the absolute value of the Hamiltonian, the existence of twisted KMS functionals and the existence of the twisted center is found. In case G is abelian and the Gibbs type condition is satisfied, the KMS states are explicitly computed in terms of characters of G.

As an application in mathematical physics, the extended field algebra of the Ising QFT is shown to be a $\mathbb{Z}_2$-crossed product of a CAR algebra which has a unique KMS state.

Chapter 1

Introduction

KMS states of C^*-dynamical systems provide on the one hand the general formalization of thermal equilibrium states [BR97], and are on the other hand natural generalizations of tracial states that are intimately connected to Tomita-Takesaki modular theory [Tak03]. Notwithstanding their importance in physics and mathematics, it can be quite difficult to decide about existence and uniqueness questions for KMS states for a given C^*-dynamical system $(\mathcal{B}, \alpha)$ consisting of a C^*-algebra $\mathcal{B}$ and dynamics (automorphic $\mathbb{R}$-action) α, i.e. to determine whether the system allows for thermal equilibrium at a given temperature, or to decide whether several pure thermodynamical phases exist.

This is also true when relevant information about a subsystem is given, i.e. when an α-invariant C^*-subalgebra $\mathcal{A} \subset \mathcal{B}$ and its KMS states are specified: KMS states of $\mathcal{A}$ might extend uniquely, non-uniquely, or not at all to KMS states of $\mathcal{B}$.

A special situation of interest arises when the inclusion $\mathcal{A} \subset \mathcal{B}$ is given by a crossed product, i.e. when $\mathcal{B} = \mathcal{A} \rtimes_\gamma G$ is the crossed product of $\mathcal{A}$ by the action γ of a discrete group G. Such a G-action will be called twist in the following. (See [Wil07] for a general account of C^*-crossed products, and [DKR66; Ara+77; AE83] for some early applications in mathematical physics.) In this context, the natural question is about the extension of a KMS state ω from $\mathcal{A}$ to its crossed product $\mathcal{A} \rtimes_\gamma G$. In this situation, the KMS condition is often of no immediate computational advantage. One rather faces a situation in which the dynamics α, the twist γ, the structure of the group G and the structure of the C^*-algebra $\mathcal{A}$ interact in a non-trivial manner.

The nature of the extension problem for KMS states depends in particular on how the dynamics is chosen. In this thesis, we will always start with a dynamics α on $\mathcal{A}$ which commutes with the twist γ. Then α extends naturally to $\mathcal{A} \rtimes_\gamma G$ by acting pointwise; this extended action acts trivially on G. From the point of view of physics, this situation amounts to enlarging the observable algebra $\mathcal{A}$ of a system by elements that are invariant under the dynamics, and ask for the equilibrium states of the enlarged system. Every KMS state on the crossed product then restricts to a

KMS state on $\mathcal{A}$.

A different and in some sense opposite choice of dynamics has also been studied in the literature: Instead of starting from a dynamics on $\mathcal{A}$, one defines a dynamics on $\mathcal{B} = \mathcal{A} \rtimes_\gamma G$ from a 1-cocycle of G, which acts non-trivial on G but trivial on $\mathcal{A}$. This construction has been investigated in particular in the context of groupoids by Neshveyev [Nes14], and generalized to so-called α-regular inclusions $\mathcal{A} \subset \mathcal{B}$ by Christensen and Thomsen [CT21]. As the dynamics is trivial on $\mathcal{A}$, KMS states of $\mathcal{B}$ restrict to (special) traces on $\mathcal{A}$. The two papers mentioned above describe the extension problem for traces on $\mathcal{A}$ to KMS states on $\mathcal{B}$, and provide various conditions under which the extension is unique. In the book of Thomsen [Tho23, Chapt. 7], also a dynamics on $\mathcal{A} \rtimes_\gamma G$ that combines a dynamics on $\mathcal{A}$ with a cocycle is studied, and a method to construct KMS weights on the crossed product starting from KMS weights on $\mathcal{A}$ is presented. We also refer to [LN03; Urs21; NS22; LZ24] for more related recent work, partially restricted to the case of traces rather than general KMS states.

Before introducing our approach to KMS states on crossed products, we want to stress the physical importance of both KMS states and crossed products. The operator algebraic approach to quantum physics associates to a physical system an algebra of observables $\mathcal{A}$ and a time evolution α. A KMS state ω at inverse temperature β on this dynamical system then describes one thermal equilibrium state of the system. KMS states therefore bridge the mathematical discipline of operator algebras with the physical theory of quantum statistical mechanics. They moreover generalize the well-known Gibbs states of quantum statistical mechanics. In particular, the state

$$\omega^{\text{Gibbs}}(\cdot) = \frac{1}{Z}\,\mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}\,\cdot)$$

is a KMS state at inverse temperature βÿ of the dynamical system $(\mathcal{B}(\mathcal{H}), \alpha)$, where the dynamics α is induced by the Hamiltonian H on $\mathcal{H}$, where it is assumed that $e^{-\beta H}$ is a trace-class operator. This trace-class condition can be weakened by focusing on an algebra of observables $\mathcal{A}$ smaller than $\mathcal{B}(\mathcal{H})$ and generalizing Gibbs states to KMS states. This allows KMS states to be directly evaluated in the thermodynamic limit, whereas Gibbs states are usually computed in systems of finite size. We discuss Gibbs states and Gibbs representations in combination with crossed products in Lemma 4.1.4.

A given physical system might allow for more than one thermal equilibrium phase at a given temperature. Assuming that the system has multiple phases, the corresponding algebra of observables will then also carry multiple KMS states at this temperature. The KMS states (at a fixed temperature) form a simplex whose extreme points can then be interpreted as the pure thermodynamical phases of the system. Furthermore, it is possible to detect phase transitions and study thermal properties in this operator algebraic framework. In [AB83; AE83; Ara84], the thermodynamic behavior of (quantum and classical) spin systems are studied within this framework, in combination with the transfer matrix method.

An illustrating example of this framework is the following: Consider a (quantum) spin system with two extremal KMS states with different magnetic properties at a given temperature T. One KMS state might have positive and the other negative average magnetization. The convex combination of these is then a KMS state with vanishing average magnetization. At a different temperature $\tilde{T} > T$, the system might only allow for a single KMS state, which moreover has vanishing average magnetization. Thus, a phase transition occurrs at a critical temperature T_c from a magnetic to an unmagnetic phase.

The algebra of observables $\mathcal{A}$ of a given quantum system typically carries a large number of symmetries $\mathrm{Aut}(\mathcal{A})$, even when restricting to the symmetries compatible with a given dynamics α. This is the starting point of the crossed product construction. The physical idea is to enlarge the observable algebra by a subgroup G of its symmetries. One therefore constructs a new algebra, called the crossed product $\mathcal{A} \rtimes G$, which includes the original observables $\mathcal{A}$ and the symmetries G. This construction works (in the C^*-setting) well in case the group G of symmetries is discrete. In case G is only locally compact, then the algebra $\mathcal{A}$ and the group G are only contained in $\mathcal{A} \rtimes_\gamma G$ in a "smeared" way. The idea to incorporate symmetries into the observable algebra can be applied in a variety of different physical settings. One application is to relate the thermal behavior of systems, which are related by symmetry transformations. This idea will be applied in Chapter 6 to the Ising QFT model. Moreover, the Fermionic and Pauli algebra of a spin system are related via the Jordan-Wigner transformation and a crossed product construction, see [AE83].

A number of recent articles was published on the coupling of QFT to gravity [Cha+23; Few+24; KL24]. In combining the algebraic approach to QFT with gravity, the need for quantum reference frames and quantum measurement schemes arises. The joint algebra of the quantum field and reference frame observables can then be shown to be a (von Neumann) crossed product. One is then typically interested in questions of entropy, which are directly related to traces, KMS states and the types of algebras arising in these constructions. These recent developments show that the idea of enlarging an operator algebra by (a subgroup) of its symmetries is a promising approach in understanding the interplay between QFT, measurement theory and gravity.

Motivated by the desire to have a criterion for extensions of KMS states to crossed products (w.r.t. the canonically extended dynamics) that can be efficiently checked in relevant examples, we here propose another point of view on the extension problem. We characterize the KMS states on the crossed product in terms of families of γ-twisted KMS functionals as they appear by restriction of a KMS state on the crossed product to the fiber over a fixed group element.

In this thesis, we restrict ourselves to crossed products that are simple on the group side in the following sense. We choose to deal with finite groups as these are always compact and amenable. A consequence of the amenability is in particular that the crossed product carries a unique C^*-norm in the sense that the universal and reduced norm coincide, which makes the distinction between the two obsolete in this thesis [Wil07, Chap. 7]. On the side of the algebra, we allow for general (unital) C^*-algebras $\mathcal{A}$ and study the extension problem for KMS states from $\mathcal{A}$ to $\mathcal{A} \rtimes G$.

In this case, a γ-twisted KMS functional $\rho : \mathcal{A} \to \mathbb{C}$ satisfies by definition

$$\rho(a\alpha_{i\beta}(b)) = \rho(b\gamma(a)), \qquad a, b \in \mathcal{A}_\alpha,$$

in standard notation (see Def. 3.1.2). Such "super" KMS functionals have been studied in the context of supersymmetry [JLW89; Kas89; BL99; Hil15]. For us, they play the role of an auxiliary object to describe the untwisted KMS states of the crossed product.

A similar role play certain operator-valued states $\varphi : G \to \mathfrak{M}_\omega$, where $\mathfrak{M}_\omega$ is the enveloping von Neumann algebra of $\mathcal{A}$ in the GNS representation of the KMS state ω. The definition of a γ-inner state is given in Definition 4.5.3. By the non-commutative Radon-Nikodým derivative, these $\mathfrak{M}_\omega$-valued states are related to (certain) families of twisted KMS functionals. These states moreover play a role in characterizing the center of the von Neumann crossed product $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ and form a bridge between Tomita-Takesaki modular theory and innerness of group actions. They have been considered in the context of traces on crossed products of non-commutative C^*-algebras [Urs21].

The general introduction to dynamical systems, crossed products and KMS states is contained in the Chapters 2 and 3. In Chapter 2, we recall the notions of dynamical systems and covariant representations, we moreover introduce their respective finitely-twisted versions. This allows us to view the crossed product itself as a dynamical system. Furthermore, we discuss the structure inclusion ι and the conditional expectation E that a crossed product by a finite (or discrete) group is naturally equipped with.

KMS states are then introduced in Chapter 3. We first recall two equivalent definitions of KMS states and twisted KMS functionals, before discussing the representation theory of KMS states and the relation to Tomita-Takesaki modular theory. Lastly, we focus on the set of KMS states $\mathcal{S}_\beta(\mathcal{A}, \alpha)$ rather than a single one.

The general analysis of KMS states on crossed products is contained in Chapter 4. As we show in Section 4.1, only the twist-invariant KMS states ω have extensions to the crossed product. One such extension is the canonical one $\hat{\omega}^{\mathrm{can}}$ which utilizes the conditional expectation. After constructing examples of crossed products with multiple KMS states using Gibbs representations, we discuss the GNS representation of the canonical extension $\hat{\omega}^{\mathrm{can}}$ in Section 4.2. The center of the enveloping von Neumann algebra in the GNS representation of $\hat{\omega}^{\mathrm{can}}$ then describes the extensions of ω to the crossed product.

We take another angle on the extension question in Section 4.3. The extensions of a KMS state ω of $\mathcal{A}$ to the crossed product can be fully characterized in terms of the positively-compatible covariant families of twisted KMS functionals (Thm. 4.3.1). In this approach, the twisted functionals encode in how many different ways a KMS state ω on $\mathcal{A}$ extends to a KMS state on $\mathcal{A} \rtimes_\gamma G$, with the trivial choice corresponding to the canonical extension $\hat{\omega}^{\mathrm{can}}$.

In Section 4.4 and 4.5, we focus on the twist $\overline{\gamma}$ on $\mathfrak{M}_\omega$ in the GNS representation. The twisted center

$$\mathcal{Z}(\mathfrak{M}_\omega, \gamma_s) := \{x \in \mathfrak{M}_\omega \mid xy = \overline{\gamma}_s(y)x \,\forall y \in \mathfrak{M}_\omega\}$$

is introduced as a starting point of the analysis. In Proposition 4.4.5 it is shown that the twisted center is related to the the modular conjugation of the standard pair $(\mathfrak{M}_\omega, \Omega)$. This allows us to show that covariant $\mathfrak{M}_\omega$-valued states on G with pointwise image in the twisted center are in bijection with the extensions of ω (Thm. 4.5.6). There are two extremal cases to consider: If γ is weakly inner (i.e. inner on $\mathfrak{M}_\omega$) and ω extremal, the extensions of ω are labeled by the characters of G. If, on the other hand, $\overline{\gamma}$ is properly outer on $\mathfrak{M}_\omega$ (i.e. outer in the factor case), then ω has a unique extension. The general situation is however more complicated, as there exists a Kallman splitting for every group element which is not multiplicative in general (Prop. 4.4.8). We analyze the relation between the structure of the group and the $\overline{\gamma}$-inner states in more detail in Section 4.6. Particularly nice results follow when restricting to simple and abelian groups (Cor. 4.6.8).

In many situations in relativistic quantum physics the dynamics has specific asymptotic properties. We therefore investigate the relation between weakly asymptotically abelian dynamics and the crossed product construction in Section 4.7. The connection of graded asymptotically abelian dynamics with twisted KMS functionals has previously been studied by Buchholz and Longo [BL99]. In line with their work, we show that in the case of asymptotically abelian dynamics and non-trivial twist, an extremal and γ-invariant KMS state ω has a unique extension (Cor. 4.7.6).

The results described so far provide a satisfactory analysis of the extension problem for KMS states on crossed products in an abstract setting. Complementing this, we show in Chapter 5 how our criterion can be checked efficiently by spectral analysis in concrete situations of interest in physics. To this end, we specialize to a particular type of C^*-dynamical system, given by the (self-dual) CAR algebra $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ over a Hilbert space with anti-unitary involution. We consider dynamics and twist implemented by Bogoliubov automorphisms, i.e. by a unitary one-parameter group $(e^{itH})_{t\in\mathbb{R}}$ on $\mathcal{K}$ and a commuting unitary representation V of G.

Physically, the CAR algebra is the observable algebra of a fermionic system with an arbitrary and non-fixed number of particles. The specialization to Bogoliubov automorphisms describes a situation, in which the dynamics and twist is given by the one-particle structure rather than the interaction between the different particles. The unique KMS state of the CAR algebra as well its GNS representation is well-known [Ara71]. We give a brief physical interpretation of the GNS representation in terms of particles and thermal holes in Chapter 5.

We specialize to the case without zero modes ($\ker H = \{0\}$) and completely determine all twisted KMS functionals for twists of finite order. It turns out that the Gibbs type condition

$$\mathrm{Tr}_{\mathcal{K}_\perp}(e^{-|\beta H_\perp|}) < \infty$$

(with $\mathcal{K}_\perp$ the orthogonal complement of $\mathrm{Eig}_1(V)$, and $H_\perp = H|_{\mathcal{K}_\perp}$ the "odd" part of the Hamiltonian) holds if and only if there exists a non-vanishing twisted KMS functional of $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ dominated by ω. In case G is abelian or the eigenvalues of H are non-degenerate, we fully understand the KMS states on the crossed product (Sec. 5.2). All extensions are explicitly determined in Theorem 5.2.3. We hint at a physical interpretation of these extensions in the outlook of this thesis.

In Chapter 6 we explain how our results apply in examples from mathematical physics. In particular, we show that the extended field algebra of the Ising QFT has the structure of a $\mathbb{Z}_2$-crossed product with a CAR algebra, and has a unique KMS state at each inverse temperature.

During the writing of this Master's Thesis, the article [SGL24] on KMS states of $\mathbb{Z}_2$-crossed products was written in co-authorship with Prof. Gandalf Lechner and Dr. Ricardo Correa da Silva. This thesis is an extended version of aforementioned article. It includes in particular an introduction to crossed products by finite groups as well as a chapter on the theory of KMS states. Moreover, the analysis of KMS states on crossed products is extended from $\mathbb{Z}_2$ to arbitrary finite groups in this thesis.

Chapter 2

Crossed Products by Finite Groups

An introduction to (finitely-)twisted dynamical systems $(\mathcal{A}, \alpha, \gamma)$ and their covariant representations (π, U, V) is the starting point of this chapter. A finitely-twisted dynamical system then allows for the construction of a crossed product $\mathcal{A} \rtimes_\gamma G$ as a dynamical system. This algebra moreover carries by construction a dynamics-equivariant structure inclusion $\iota : \mathcal{A} \hookrightarrow \mathcal{A} \rtimes_\gamma G$ and conditional expectation $E : \mathcal{A} \rtimes_\gamma G \twoheadrightarrow \mathcal{A}$.

The crossed product $\mathcal{A} \rtimes_\gamma G$ constructed in this chapter can be thought of as the algebra generated by the observable algebra $\mathcal{A}$ together with the group of symmetries $G \subset \mathrm{Aut}(\mathcal{A})$ as unitary elements. The dynamics $\hat{\alpha}$ on $\mathcal{A} \rtimes_\gamma G$ extends the dynamics α on $\mathcal{A}$ and leaves the group G pointwise invariant due to the commutation relation $\alpha \circ \gamma = \gamma \circ \alpha$.

Although the group G is an important datum for a twisted dynamical system, it will be suppressed in the notation. After giving the general definition of a twisted C^*- or W^*-dynamical system, we will mostly work with finitely-twisted version where the topological group G is finite. We will moreover assume here and in the following that the C^*-algebra $\mathcal{A}$ is unital.

Definition 2.0.1. *A twisted C^*-dynamical system is a triple $(\mathcal{A}, \alpha, \gamma)$ consisting of a C^*-algebra $\mathcal{A}$, a strongly continuous automorphic $\mathbb{R}$-action $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathcal{A}$, and a strongly continuous automorphic G-action $\gamma : G \to \mathrm{Aut}\,\mathcal{A}$ by a topological group G satisfying*

$$\alpha_t \circ \gamma_s = \gamma_s \circ \alpha_t, \qquad t \in \mathbb{R}, s \in G. \tag{2.0.1}$$

A twisted W^*-dynamical system is a triple $(\mathfrak{M}, \alpha, \gamma)$ consisting of a von Neumann algebra $\mathfrak{M}$, a weakly continuous automorphic $\mathbb{R}$-action $\alpha : \mathbb{R} \to \mathrm{Aut}\,\mathfrak{M}$, and a weakly continuous automorphic G-action $\gamma : G \to \mathrm{Aut}\,\mathfrak{M}$ by a topological group G satisfying equation (2.0.1)*.*

A twisted C^*- or W^*-dynamical system $(\mathcal{A}, \alpha, \gamma)$ is called a finitely-twisted C^*- or W^*-dynamical if the topological group G is finite.

Oftentimes the distinction between (finitely-)twisted C^*- and W^*-dynamical systems is not necessary. In these cases we will only speak of a (finitely-)twisted dynamical system $(\mathcal{A}, \alpha, \gamma)$. When the twist γ is absent or not relevant, we refer to $(\mathcal{A}, \alpha)$ as a dynamical system and similarly to $(\mathcal{A}, \gamma)$ as a (finitely-)twisted system. The action α is also called the dynamics of the system, and γ is called the twist. As usual, we write for a $\mathcal{A}_\alpha \subset \mathcal{A}$ for the norm dense *-subalgebra of entire analytic elements for a C^*-dynamical system $(\mathcal{A}, \alpha)$. Similarly, $\mathfrak{M}_\alpha$ is the weakly dense *-subalgebra of entire analytic elements for a W^*-dynamical system, see A.5.

The commutation condition (2.0.1) allows one to view a (finitely-)twisted dynamical system $(\mathcal{A}, \alpha, \gamma)$ as a dynamical system $(\mathcal{A}, \alpha \times \gamma)$. We will however not take this viewpoint, as we treat $\mathbb{R}$ and G differently in the crossed product construction.

Definition 2.0.2. *A covariant representation of a twisted C^*-dynamical system $(\mathcal{A}, \alpha, \gamma)$ is a triple (π, U, V) consisting of a non-degenerate representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and two unitary representations $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ and $V : G \to \mathcal{U}(\mathcal{H})$ on the same Hilbert space satisfying*

$$\pi(\alpha_t(a)) = U_t \pi(a) U_t^*, \quad \pi(\gamma_s(a)) = V_s \pi(a) V_s^* \quad \text{and} \quad U_t V_s = V_s U_t.$$

Let (π, V) be a covariant representation of the finitely-twisted system $(\mathcal{A}, \gamma)$. Then

$$\pi \rtimes V(f) := \sum_{s \in G} \pi(f(s)) V_s, \quad \forall f \in C(G, \mathcal{A}),$$

is called the integrated form of (π, V).

The integrated form of a covariant representation (π, U, V) of a finitely-twisted C^*-dynamical system $(\mathcal{A}, \alpha, \gamma)$ is similarly taken to be $\pi \rtimes V$. Note that α and γ play different roles in the integrated form. Although the integrated form of a covariant representation (π, V) of a twisted C^*-algebra $(\mathcal{A}, \gamma)$ can be formulated for a locally compact group G and $f \in C_c(G, \mathcal{A})$ [Wil07, Chap. 2.3], we will not discuss this definition here as we will mostly deal with finite groups G in the following.

Dynamical systems are the foundation for the construction of crossed product algebras. This construction can be done in broad generality for a locally compact group G acting on a C^*- or von Neumann algebra, see [Tak67; BR87; Wil07]. We restrict ourselves however to the context of a finite group G and are interested in the crossed product algebra as a dynamical system. This is in particular the case as $\mathcal{A}$ can be viewed as a subalgebra of the crossed product for G finite.

The construction of the crossed product $\mathcal{A} \rtimes_\gamma G$ is as follows. For a finitely-twisted C^*-dynamical system $(\mathcal{A}, \alpha, \gamma)$ consider the set of functions $C(G, \mathcal{A})$. This space is a unital $*$-algebra with the convolution product, involution and unit

$$f * g(s) := \sum_{r\in G} f(r)\gamma_r(g(r^{-1}s)), \quad f^*(s) := \gamma_s(f(s^{-1})^*), \quad \mathbb{1} := \delta_e. \tag{2.0.2}$$

This space carries a unique C^*-norm as G is assumed to be finite. One way of constructing the C^*-norm is via induction of representations. Take a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ and consider the corresponding Hilbert space $L^2(G, \mathcal{H}_\pi)$ with scalar product

$$\langle \psi, \varphi \rangle = \sum_{s\in G} \langle \psi(s), \varphi(s) \rangle, \quad \psi, \varphi \in L^2(G, \mathcal{H}_\pi).$$

This space carries a covariant representation $(\hat{\pi}, V)$ of $(\mathcal{A}, \gamma)$ by

$$(\hat{\pi}(a)\psi)(s) = \pi(\gamma_s^{-1}(a))\psi(s) \quad \text{and} \quad (V_r\psi)(s) = \psi(r^{-1}s). \tag{2.0.3}$$

The integrated form $\hat{\pi} \rtimes V$ is then a faithful *-representation of $C(G, \mathcal{A})$ in $\mathcal{B}(L^2(G, \mathcal{H}_\pi))$ [Wil07, Lemma 2.26]. Pulling back the C^*-norm from $\mathcal{B}(L^2(G, \mathcal{H}_\pi))$ yields a C^*-norm $\|\cdot\|$ on $C(G, \mathcal{A})$, where no closure process is necessary due to G being finite.

The above constructed C^*-algebra is denoted $\mathcal{A} \rtimes_\gamma G$ and called the crossed product of $\mathcal{A}$ by G. More specifically, this construction is used for the definition of the reduced crossed product of a C^*-dynamical system, see [Wil07, Definition 7.7]. As every finite group is amenable, there is no need to distinguish between the reduced and universal crossed product [Wil07, Theorem 7.13].

Note that the C^*-norm $\|\cdot\|$ is dominated by the L^1-norm $\|\cdot\|_1$. Take a faithful representation π of $\mathcal{A}$ and consider $\varphi, \psi \in L^2(G, \mathcal{H}_\pi)$ normalized and $f \in \mathcal{A} \rtimes_\gamma G$. Then

$$|\langle \psi, \hat{\pi} \rtimes V(f)\varphi \rangle| \le \sum_{s\in G} |\langle \psi, \hat{\pi}(f(s))V_s\varphi \rangle| \le \sum_{s\in G} \|\psi\| \|f(s)\| \|\varphi\| = \sum_{s\in G} \|f(s)\| = \|f\|_1$$

by Cauchy-Schwarz inequality. Taking the supremum over $\varphi, \psi \in L^2(G, \mathcal{H}_\pi)$ normalized yields $\|f\| \le \|f\|_1$.

It is also clear that $\mathcal{A}$ is faithfully embedded in $\mathcal{A} \rtimes G$ by the inclusion

$$\iota : \mathcal{A} \hookrightarrow \mathcal{A} \rtimes_\gamma G, \quad a \mapsto a \cdot \delta_e, \tag{2.0.4}$$

as G is a finite and thus discrete group. With the faithful conditional expectation there exists a further structure map

$$E : \mathcal{A} \rtimes_\gamma G \twoheadrightarrow \mathcal{A}, \quad f \mapsto f(e) \tag{2.0.5}$$

satisfying $E \circ \iota = \mathbb{1}_{\mathcal{A}}$.

As $\mathcal{A}$ carries a dynamics α which is compatible with the G-action γ, also $\mathcal{A} \rtimes_\gamma G$ carries a natural dynamics as well by defining

$$(\hat{\alpha}_t f)(s) := \alpha_t(f(s)). \tag{2.0.6}$$

The following calculations show that $\hat{\alpha}$ is a dynamics on $\mathcal{A} \rtimes_\gamma G$. Firstly, $\hat{\alpha}_t \in \mathrm{Aut}(\mathcal{A} \rtimes G)$ by

$$\begin{aligned}
(\hat{\alpha}_t(f * g))(s) &= \alpha_t(f * g(s)) = \sum_{r\in G} \alpha_t(f(r)\gamma_r(g(r^{-1}s))) \\
&= \sum_{r\in G} \alpha_t(f(r))\gamma_r(\alpha_t(g(r^{-1}s))) = (\hat{\alpha}_t(f) * \hat{\alpha}_t(g))(s), \\
(\hat{\alpha}_t(f^*))(s) &= \alpha_t(f^*(s)) = \alpha_t(\gamma_s(f(s^{-1})^*)) = \gamma_s(\alpha_t(f(s^{-1}))^*) = (\hat{\alpha}_t(f)^*)(s).
\end{aligned}$$

Furthermore, $\hat{\alpha}$ is a strongly continuous 1-parameter group

$$\begin{aligned}
(\hat{\alpha}_{t+\tau} f)(s) &= \alpha_{t+\tau}(f(s)) = \alpha_t(\alpha_\tau(f(s))) = ((\hat{\alpha}_t \circ \hat{\alpha}_\tau)f)(s), \\
\|\hat{\alpha}_t(f) - f\| &\le \|\hat{\alpha}_t(f) - f\|_1 = \sum_{s\in G} \|\alpha_t(f(s)) - f(s)\| \to 0,
\end{aligned}$$

where $\|\cdot\|_1$ denotes the L^1-norm on $C(G, \mathcal{A})$. Clearly, $(\mathcal{A}, \alpha)$ can be embedded into $(\mathcal{A} \rtimes G, \hat{\alpha})$ via ι as a C^*-dynamical system and E is an $\mathbb{R}$-equivariant conditional expectation. Furthermore, the set of functions $f : G \to \mathcal{A}_\alpha$ with range in the analytic elements of $\mathcal{A}$ form a dense subalgebra of the analytic elements of $\mathcal{A} \rtimes_\gamma G$ for $\hat{\alpha}$.

Let us collect these findings in the following lemma.

Lemma 2.0.3. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system. Then the C^*-crossed product $(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ is a C^*-dynamical system,*

$$\iota : \mathcal{A} \hookrightarrow \mathcal{A} \rtimes G, \quad a \mapsto a \cdot \delta_e,$$

is a faithful morphism of C^*-dynamical systems and

$$E : \mathcal{A} \rtimes G \twoheadrightarrow \mathcal{A}, \quad f \mapsto f(e)$$

is an $\mathbb{R}$-equivariant faithful conditional expectation satisfying $E \circ \iota = \mathbb{1}_{\mathcal{A}}$.

A direct consequence of the above construction is the following corollary.

Corollary 2.0.4. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system and (π, U, V) a covariant representation. Then $(\pi \rtimes V, U)$ is a covariant representation of $(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$.*

Proof. Take $f \in \mathcal{A} \rtimes_\gamma G$ and compute

$$\begin{aligned}
\pi \rtimes V(\hat{\alpha}_t(f)) &= \sum_{s\in G} \pi((\hat{\alpha}_t f)(s))V_s = \sum_{s\in G} \pi(\alpha_t(f(s)))V_s = \sum_{s\in G} U_t \pi(f(s))U_t^* V_s \\
&= U_t \sum_{s\in G} \pi(f(s))V_s U_t^* = U_t(\pi \rtimes V(f))U_t^*.
\end{aligned}$$

□

The crossed product construction can be done analogously for a finitely-twisted W^*-dynamical system $(\mathfrak{M}, \alpha, \gamma)$. The algebraic definitions of the convolution, involution and unit in equation (2.0.2) as well as the definition of the dynamics $\hat{\alpha}$ in equation (2.0.6) are exactly the same. It remains to be checked that the W^*-crossed product $\mathfrak{M} \rtimes_\gamma G$ is again a von Neumann algebra. Consider again $L^2(G, \mathcal{H})$ and the integrated form of (2.0.3), where $\pi : \mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ and take a net $f_i \in \mathfrak{M} \rtimes_\gamma G$ which is SOT-convergent in $\mathcal{B}(L^2(G, \mathcal{H}))$. Then for $s \in G, v \in \mathcal{H}$ and $\psi_v(r) = v\delta_e(r)$ it follows

$$\left\|\gamma_s^{-1}((f_i - f_j)(s))v\right\|^2 \leq \sum_{s\in G} \left\|\gamma_s^{-1}((f_i - f_j)(s))v\right\|^2 = \|\hat{\pi} \rtimes V(f_i - f_j)\psi_v\|^2 \to 0.$$

By weak continuity of γ, $f_i(s)$ is a convergent net in $\mathfrak{M}$. Denote its limit by x_s, then f_i converges to $f \in \mathfrak{M} \rtimes_\gamma G$, where $f(s) = x_s$.

$$\begin{aligned}\|\hat{\pi} \rtimes V(f_i - f)\psi\|^2 &= \sum_{s\in G} \|(\hat{\pi} \rtimes V(f_i - f)\psi)(s)\|^2 \\ &= \sum_{s,r\in G} \left\|\gamma_r^{-1}(f_i(r) - x_r)\psi(r^{-1}s)\right\|^2 \to 0.\end{aligned}$$

This shows that the reduced crossed product construction defines a von Neumann algebra $\mathfrak{M} \rtimes_\gamma G$. Similarly, the dynamics $\hat{\alpha}$ is weakly continuous by

$$\|\hat{\pi} \rtimes V(\hat{\alpha}_t(f) - f)\psi\|^2 = \sum_{s,r\in G} \left\|\gamma_r^{-1}(\alpha_t(f(r)) - f(r))\psi(r^{-1}s)\right\|^2 \to 0.$$

The inclusion ι defined in equation (2.0.4) is an inclusion of W^*-dynamical systems and the $\mathbb{R}$-equivariant faithful conditional expectation E defined in (2.0.5) is normal.

This proves the following lemma.

Lemma 2.0.5. *Let $(\mathfrak{M}, \alpha, \gamma)$ be a finitely-twisted W^*-dynamical system. Then the W^*-crossed product $(\mathfrak{M} \rtimes_\gamma G, \hat{\alpha})$ is a W^*-dynamical system and*

$$\iota : \mathfrak{M} \hookrightarrow \mathfrak{M} \rtimes_\gamma G, \quad a \mapsto a \cdot \delta_e,$$

is a faithful morphism of W^*-dynamical systems and

$$E : \mathfrak{M} \rtimes_\gamma G \twoheadrightarrow \mathfrak{M}, \quad f \mapsto f(e)$$

is an $\mathbb{R}$-equivariant faithful normal conditional expectation satisfying $E \circ \iota = \mathbb{1}_{\mathfrak{M}}$.

Chapter 3

KMS States and their Representation Theory

This chapter contains an introduction to KMS states and their representation theory. We show that KMS states are invariant under the dynamics, which allows us to physically interpret (extremal) KMS states as thermal equilibrium phases. Similarly, twisted KMS functionals are invariant under the dynamics and twist. Therefore, they are a suitable candidate for a notion of thermal equilibrium in supersymmetric theories. A more elaborate discussion can be found below in Section 3.1. We moreover introduce the basics of Tomita-Takesaki modular theory, as it is in close connection to KMS states on von Neumann algebras. Lastly, we shift the focus to the structure of the set of KMS states in Section 3.4 and briefly comment on the interpretation of extremal KMS states as pure thermodynamic phases.

3.1 Introduction to KMS States

We begin this section with the introduction to KMS states and twisted KMS functionals on an algebra of observables. This introduction is then followed by a discussion of their physical interpretation and their relation to thermodynamics as thermal equilibrium states.

For a dynamical system $(\mathcal{A}, \alpha)$, we adopt the physics definition of KMS states and γ-twisted KMS functionals [BL99]. As a preparation for the definition we introduce the notation

$$S_\beta := \begin{cases} \{z \in \mathbb{C} \,|\, 0 < \mathrm{Im}\{z\} < \beta\} & \forall \beta \geq 0, \\ \{z \in \mathbb{C} \,|\, \beta < \mathrm{Im}\{z\} < 0\} & \forall \beta < 0, \end{cases}$$

and $\overline{S}_\beta$ for the closure of S_β for $\beta \neq 0$. We further set $\overline{S}_0 := \mathbb{R}$ and note that $S_0 = \emptyset$.

The set of states of a C^*- or W^*-algebra $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A})$. Furthermore, we introduce the definition of a normal state ω for a concretely represented von Neumann algebra $\mathfrak{M}$. We refer to [BR87, Thm. 2.4.21] for the equivalence of this definition with the characterization in terms of least upper bounds.

Definition 3.1.1. *Let $\mathfrak{M} \subset \mathcal{B}(\mathcal{H})$ be a concrete von Neumann algebra and $\omega \in \mathcal{S}(\mathfrak{M})$. Then ω is called normal if there exists a positive trace-class operator $\mathcal{D} \in \mathcal{B}(\mathcal{H})$ with $\mathrm{Tr}(\mathcal{D}) = 1$ s.t.*

$$\omega(x) = \mathrm{Tr}(\mathcal{D}x), \quad \forall x \in \mathfrak{M}.$$

Note that every unit vector $\Omega \in \mathcal{H}$ induces a normal state ω_Ω by $\omega_\Omega(x) = \langle \Omega, x\Omega \rangle = \mathrm{Tr}(\mathcal{D}_\Omega x)$, where $\mathcal{D}_\Omega = |\Omega\rangle \langle \Omega|$.

Definition 3.1.2. *Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system. A state $\omega \in \mathcal{S}(\mathcal{A})$ is called α-KMS state at inverse temperature $\beta \in \mathbb{R}$ or (α, β)-KMS state if for all $a, b \in \mathcal{A}$ there exists a function $F_{a,b} : \overline{S}_\beta \to \mathbb{C}$ which is analytic on S_β and bounded and continuous on $\overline{S}_\beta$ s.t.*

$$\begin{aligned} F_{a,b}(t) &= \omega(a\alpha_t(b)), \\ F_{a,b}(t + i\beta) &= \omega(\alpha_t(b)a) \end{aligned}$$

for all $t \in \mathbb{R}$. We denote by $\mathcal{S}_\beta(\mathcal{A}, \alpha)$ the set of (α, β)-KMS states.

If $(\mathfrak{M}, \alpha)$ is a W^*-dynamical system, then a state $\omega \in \mathcal{S}(\mathfrak{M})$ is called (α, β)-KMS state if ω is a normal and a KMS state in the above sense.

Let further be $\gamma \in \mathrm{Aut}\,\mathcal{A}$ (not necessarily commuting with α). A functional ρ on $\mathcal{A}$ is called γ-twisted α-KMS functional at inverse temperature $\beta \in \mathbb{R}$ or γ-twisted (α, β)-KMS functional if it is continuous and for all $a, b \in \mathcal{A}$ there exists a function $G_{a,b} : \overline{S}_\beta \to \mathbb{C}$ which is analytic on S_β and bounded and continuous on $\overline{S}_\beta$ s.t.

$$\begin{aligned} G_{a,b}(t) &= \rho(a\alpha_t(b)), \\ G_{a,b}(t + i\beta) &= \rho(\alpha_t(b)\gamma(a)) \end{aligned}$$

for all $t \in \mathbb{R}$. We denote by $\mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma)$ the set of γ-twisted (α, β)-KMS functionals.

In [BR97] a number of equivalent characterizations of a KMS state are given. We have chosen to give the above definition in terms of analytic functions as it is the most conceptual one. In practice however, one often works with the dense set of analytic elements $\mathcal{A}_\alpha$ (see A.5) instead of the whole algebra $\mathcal{A}$. We thus refer to the equivalent definition of a KMS state in terms of analytic elements in Def. A.6. Furthermore, note that the above definition of a γ-twisted (α, β)-KMS functional includes continuity, as the twisted KMS functionals needed for extensions of KMS states will always be bounded. Unbounded twisted KMS functionals do however have application in other contexts such as supersymmetry [JLW89; Kas89; BL99; Hil15].

It would be straightforward to define twisted KMS functionals on von Neumann algebras, we will however not need this definition in the following. In the setting of interest to us, the automorphism γ is derived from a G-action which commutes with α. This is however not necessary for the above definition. Further note that a state is a $(\alpha, 0)$-KMS state if, and only if, it is a tracial state. For $\beta = -1$ one recovers the convention of Tomita-Takesaki theory.

For later convenience for the reader, we derive the invariance of KMS states and twisted KMS functionals under the dynamics. For twisted KMS functionals we moreover show the invariance under the twist [BR97; BL99].

Lemma 3.1.3. *Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system, $\gamma \in \operatorname{Aut}\mathcal{A}$ and $\rho \in \mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma)$ for $\beta \neq 0$. Then ρ is α- and γ-invariant.*

Applying this result to $\gamma = \mathbb{1}$ and $\omega = \rho \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$ shows that every KMS state is α-invariant.

Proof. We use the analytic element version of the KMS definition here, see A.6. Firstly, we prove the γ-invariance. Take $a \in \mathcal{A}_\alpha$, it then holds

$$\rho(a) = \rho(a\alpha_{i\beta}(\mathbb{1})) = \rho(\mathbb{1}\gamma(a)) = \rho(\gamma(a)).$$

The γ-invariance follows by continuity.

The α-invariance of ρ is slightly more involved. The argument works analogous for $\beta \gtrless 0$, we thus assume $\beta > 0$. Take $a \in \mathcal{A}_\alpha$ and consider the entire analytic function

$$F : z \mapsto \rho(\alpha_z(a)).$$

This function is bounded on the strip $\overline{S}_\beta$ by

$$|\rho(\alpha_z(a))| \leq \|\rho\| \|\alpha_z(a)\| = \|\rho\| \left\|\alpha_{\Re(z)}(\alpha_{i\Im(z)}(a))\right\| = \|\rho\| \left\|\alpha_{i\Im(z)}(a)\right\| \leq \|\rho\| N,$$

where $N = \sup\{\left\|\alpha_{i\Im(z)}(a)\right\|, \Im(z) \in [0, \beta]\} < \infty$ by the continuity of $z \mapsto \|\alpha_z(a)\|$. From the twisted KMS condition it follows

$$F(z + i\beta) = \rho(\mathbb{1}\alpha_{i\beta}(\alpha_z(a))) = \rho(\alpha_z(a)\gamma(\mathbb{1})) = \rho(\alpha_z(a)) = F(z).$$

Therefore, F is entire analytic and bounded and thus constant by Liouville's theorem A.1. It directly follows $\rho(a) = \rho(\alpha_t(a))$ for $a \in \mathcal{A}$ by continuity of ρ. □

That the above defined KMS condition is a physically sensible notion of thermal equilibrium is not straightforward. As we have shown, a KMS state is necessarily invariant under the dynamics α, which is a necessary condition for a notion of equilibrium. Moreover, if a Gibbs state

$$\omega^{\text{Gibbs}}(\cdot) = \frac{1}{Z}\,\mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}\,\cdot)$$

exists for some inverse temperature β, it satisfies the KMS property. If one further considers the thermodynamic limit of Gibbs states, the KMS property still holds [BR97, Ex. 5.3.2]. This is exactly the strength of the KMS condition, as it can be directly evaluated in the thermodynamical limit. Physically, the extreme points of the set of KMS states can be interpreted as thermodynamical phases. A general KMS state can then be decomposed into a convex combination of extremal KMS states and thus describes a mixture of thermodynamic phases, as will be discussed in Section 3.4. A more elaborate discussion of KMS states can be found in [BR97, Chap. 5.3], while some of the first treatments can be found here [Kub57; MS59; HHW67].

3.2 Representation Theory of KMS States

Having introduced the concept of KMS states in the above section, we now turn to their representation theory. A first notable consequence of the KMS condition is that their GNS vector states are separating for the associated von Neumann algebra. This connects the theory of KMS states with Tomita-Takesaki modular theory.

We call a von Neumann algebra *concrete* if we want to emphasize that it is represented on some Hilbert space $\mathcal{H}$. This becomes especially important when talking about standard vectors Ω .

Definition 3.2.1. *Let $\mathcal{A}$ be a C^*- or von Neumann algebra. A (normal) state $\omega \in \mathcal{S}(\mathcal{A})$ is called separating for $\mathcal{A}$ if the corresponding annihilator ideal*

$$\mathcal{I}_\omega := \{a \in A \,:\, \omega(a^*a) = 0\}$$

is a *-ideal.

Let $\mathfrak{M}$ be a concrete von Neumann algebra. A vector $\Omega \in \mathcal{H}$ is called separating for $\mathfrak{M}$ if for $x \in \mathfrak{M}$

$$x\Omega = 0 \implies x = 0.$$

A vector $\Omega \in \mathcal{H}$ is called standard for $\mathfrak{M}$ if Ω is cyclic and separating for $\mathfrak{M}$ and $(\mathfrak{M}, \Omega)$ is called a standard pair.

As a preparation for the following proposition, we prove the following lemma.

Lemma 3.2.2. *Let* $(\mathfrak{M}, \alpha)$ *be a* W^**-dynamical system and* $\Omega \in \mathcal{H}$ *a cyclic unit vector such that the corresponding vector state* $\omega \in \mathcal{S}(\mathfrak{M})$ *is an* (α, β)*-KMS state for* $\mathfrak{M}$*.*

Then ω *is a separating state and* Ω *is standard for* $\mathfrak{M}$*.*

Proof. As ω is given by a vector, it is normal. Take $x \in \mathcal{I}_\omega$. In order to show that $\mathcal{I}_\omega$ is a *-ideal, we have to show $\omega(xx^*) = 0$. Consider the function $F : \overline{S}_\beta \to \mathbb{C}$, $t \mapsto \omega(x^* \alpha_t(x))$, which is bounded and continuous on $\overline{S}_\beta$ and analytic on S_β. It then follows by Cauchy-Schwarz inequality

$$|\omega(x^* \alpha_t(x))|^2 \leq \omega(x^* x)\omega(\alpha_t(x^* x)) = 0.$$

Using Schwarz Reflection Principle A.2 on F yields an analytic extension $\tilde{F}$ of F to $S_\beta \cup \mathbb{R} \cup (-S_\beta)$ that vanishes on the real line $\mathbb{R}$. Application of the Identity Theorem A.3 shows that $\tilde{F}$ and consequently F vanish on $\overline{S}_\beta$ by continuity. The KMS condition now implies $\omega(xx^*) = F(i\beta) = 0$. Hence, ω is separating.

It remains to be shown that Ω is separating. Take $x \in \mathfrak{M}$ satisfying $x\Omega = 0$. Then $yx\Omega = 0$ for all $y \in \mathfrak{M}$ and moreover

$$\|x^* y^* \Omega\|^2 = \|(yx)^* \Omega\|^2 = \omega(yx(yx)^*) = 0,$$

as $\omega((yx)^* yx) \leq \|y\|^2 \omega(x^* x) = \|y\|^2 \|x\Omega\|^2 = 0$ and ω is separating. This shows that x^* vanishes on $\mathfrak{M}\Omega$ and as Ω was assumed to be cyclic for $\mathfrak{M}$ it follows $x = 0$ by continuity. This proves the assertion. □

By an analogous argument it can be shown that every KMS state ω on a C^*-dynamical system $\mathcal{A}$ is separating for $\mathcal{A}$. However, the GNS vector Ω of a separating non-KMS state ω on $\mathcal{A}$ is in general not separating for the corresponding von Neumann algebra $\mathfrak{M}_\omega = \pi_\omega(\mathcal{A})''$. This is a special property of KMS states on C^*-dynamical systems, which we prove in the following [BR97, Cor. 5.3.4, Cor. 5.3.9].

Proposition 3.2.3. *Let* $(\mathcal{A}, \alpha)$ *be a* C^**-dynamical system and* $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$*. Denote by* $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ *the GNS triple and* $\mathfrak{M}_\omega = \pi_\omega(\mathcal{A})''$*.*

Then $(\mathfrak{M}_\omega, \overline{\alpha})$ *is a* W^**-dynamical system, where* $\overline{\alpha}$ *is the unique weakly continuous automorphic* $\mathbb{R}$*-action satisfying*

$$\overline{\alpha}_t(\pi_\omega(a)) = \pi_\omega(\alpha_t(a)), \quad \forall a \in \mathcal{A}, t \in \mathbb{R}.$$

The state $\overline{\omega}$ *induced by* Ω *is a* $(\overline{\alpha}, \beta)$*-KMS state for* $(\mathfrak{M}_\omega, \overline{\alpha})$*. Moreover,* Ω *is standard for* $\mathfrak{M}_\omega$*.*

Proof. Lemma 3.1.3 shows that ω is α-invariant and thus

$$U_\omega(t)\pi_\omega(a)\Omega := \pi_\omega(\alpha_t(a))\Omega$$

defines a unitary operator. The calculation

$$\|U_\omega(t)\pi_\omega(a)\Omega\|^2 = \omega(\alpha_t(a^*a)) = \omega(a^*a) = \|\pi_\omega(a)\Omega\|^2$$

shows that $U_\omega(t)$ is a densely-defined isometry with dense range $\pi_\omega(\mathcal{A})\Omega$ and hence there exists a unique continuous extension which is moreover unitary. Clearly, U_ω is a representation of $\mathbb{R}$. The strong continuity can be checked on the dense subspace $\pi_\omega(\mathcal{A})\Omega$ by

$$\|U_\omega(t)\pi_\omega(a)\Omega - \pi_\omega(a)\Omega\|^2 = \omega((\alpha_t(a)-a)^*(\alpha_t(a)-a)) \leq \|\alpha_t(a)-a\|^2 \xrightarrow{t\to 0} 0.$$

Defining $\overline{\alpha}$ on $\mathfrak{M}_\omega$ by

$$\overline{\alpha}_t(x) := U_\omega(t)xU_\omega(-t), \quad \forall x \in \mathfrak{M}_\omega$$

yields a weakly continuous $\mathbb{R}$-action satisfying

$$\overline{\alpha}_t(\pi_\omega(a)) = \pi_\omega(\alpha_t(a)) = U_\omega(t)\pi_\omega(a)U_\omega(-t)$$

on the SOT-dense subalgebra $\pi_\omega(\mathcal{A})$ of $\mathfrak{M}_\omega$. This furthermore shows $\overline{\alpha}_t(x) \in \mathfrak{M}_\omega$ for $x \in \mathfrak{M}_\omega$ as

$$\overline{\alpha}_t(x) = \overline{\alpha}_t(\text{SOT-lim}\,\pi_\omega(a_i)) = \text{SOT-lim}\,\pi_\omega(\alpha_t(a_i)) \in \mathfrak{M}_\omega,$$

where x is strongly approximated by the net $(\pi_\omega(a_i))_{i\in I}$. Therefore, $(\mathfrak{M}_\omega, \overline{\alpha})$ is a W^*-dynamical system.

By construction, $\pi_\omega(\mathcal{A}_\alpha)$ is a strongly dense subalgebra of $\mathfrak{M}_\omega$ of analytic elements for $\overline{\alpha}$. Moreover, $\overline{\omega}$ is the vector state associated to Ω and thus normal. This directly implies that $\overline{\omega} \in \mathcal{S}_\beta(\mathfrak{M}_\omega, \overline{\alpha})$.

As Ω is a cyclic vector state and the corresponding state $\overline{\omega}$ is a KMS state, Lemma 3.2.2 is applicable. Thus Ω is standard for $\mathfrak{M}_\omega$. □

3.3 Introduction to Modular Theory

The GNS vector Ω of a KMS state ω is standard for the von Neumann algebra $\mathfrak{M}_\omega$, as was shown in the section above. We now turn to the opposite viewpoint of Tomita-Takesaki theory and take as a starting point a concrete von Neumann algebra $\mathfrak{M}$ with a standard vector Ω. This allows for the definition of the Tomita operator and the recovery of the KMS condition. Our brief introduction follows [BR87, Chap. 2.5].

We begin this section by analyzing the relation between a von Neumann algebra $\mathfrak{M}$, its commutant $\mathfrak{M}'$ and a vector Ω.

Lemma 3.3.1. *Let $\mathfrak{M}$ be a concrete von Neumann algebra and $\Omega \in \mathcal{H}$. The following are equivalent:*

1) *Ω is cyclic for $\mathfrak{M}$;*

2) *Ω is separating for $\mathfrak{M}'$.*

In particular, Ω is standard for $\mathfrak{M}$ if, and only if, Ω is standard for $\mathfrak{M}'$.

Proof. 1) $\Rightarrow$ 2) : Let $x' \in \mathfrak{M}'$ and $x'\Omega = 0$. Then $0 = yx'\Omega = x'y\Omega$ for $y \in \mathfrak{M}$. Hence x' vanishes on the dense subspace $\mathfrak{M}\Omega$ and $x' = 0$ by continuity.

2) $\Rightarrow$ 1) : Consider the projection p onto $\overline{\mathfrak{M}\Omega}$. Then $\overline{\mathfrak{M}\Omega}$ and $\overline{\mathfrak{M}\Omega}^{\perp}$ are subrepresentations of $\mathfrak{M}$. This shows that $p \in \mathfrak{M}'$, since for $\varphi = \varphi_p \oplus \varphi_{p^\perp} \in \mathcal{H}$

$$xp\varphi = x\varphi_p = p(x\varphi_p) = p(x\varphi_p \oplus x\varphi_{p^\perp}) = px\varphi.$$

Using that Ω is separating for $\mathfrak{M}'$ together with $p\Omega = \Omega$ yields $p = \mathbb{1}$. This shows that Ω is cyclic for $\mathfrak{M}$.

Application of the above results to $\mathfrak{M} = \mathfrak{M}''$ yields the remaining assertion. □

This result allows for the definition of the Tomita operator as a closed anti-linear operator.

Corollary 3.3.2. *Let $\mathfrak{M}$ be a concrete von Neumann algebra and $\Omega \in \mathcal{H}$ be standard. Then*

$$S_0 : \mathfrak{M}\Omega \to \mathfrak{M}\Omega, \quad x\Omega \mapsto x^*\Omega$$

is a well-defined, densely-defined, closable anti-linear involution.

Its closure - denoted by S - is called the Tomita operator of the standard pair $(\mathfrak{M}, \Omega)$.

Proof. S_0 is well-defined as Ω is separating for $\mathfrak{M}$ and clearly an anti-linear involution. Moreover, S_0 is densely-defined by the cyclicity of Ω. In order to show that S_0 is closable, consider the operator

$$F_0 : \mathfrak{M}'\Omega \to \mathfrak{M}'\Omega, \quad x'\Omega \mapsto (x')^*\Omega,$$

which is similarly a well-defined anti-linear involution with a dense domain. The calculation

$$\langle y'\Omega, S_0(x\Omega)\rangle = \langle y'\Omega, x^*\Omega\rangle = \langle x\Omega, (y')^*\Omega\rangle = \langle x\Omega, F_0(y'\Omega)\rangle = \overline{\langle F_0(y'\Omega), x\Omega\rangle}$$

shows that $\mathfrak{M}\Omega$ is contained in the domain of F_0^*. It furthermore shows that F_0^* is a closed extension of S_0 and thus S_0 is closable. □

The polar decomposition of closed anti-linear operators can thus be applied to the Tomita operator S.

Definition 3.3.3. *Let $(\mathfrak{M}, \Omega)$ be a standard pair and S be the associated Tomita operator with polar decomposition*

$$S = J\Delta^{\frac{1}{2}}.$$

The positive operator Δ is called the modular operator and the anti-unitary involution J is called the modular conjugation of $(\mathfrak{M}, \Omega)$. The pair (J, Δ) is called the modular data of $(\mathfrak{M}, \Omega)$.

These modular data satisfy a number of relations [BR87, Prop. 2.5.11],

$$\Delta = S^*S, \quad \Delta^{-1} = SS^*, \quad J = J^* = J^{-1}, \quad J\Delta^{\frac{1}{2}}J = \Delta^{-\frac{1}{2}}, \quad J\Omega = \Omega, \quad \Delta^{\frac{1}{2}}\Omega = \Omega.$$

We conclude this introductory section to modular theory by citing the Tomita-Takesaki Theorem [BR87, Thm. 2.5.14].

Theorem 3.3.4. *Let $(\mathfrak{M}, \Omega)$ be a standard pair and (J, Δ) the associated modular data.*

Then the following holds

$$J\mathfrak{M}J = \mathfrak{M}' \quad \text{and} \quad \Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}, \quad \forall t \in \mathbb{R}.$$

The Tomita-Takesaki Theorem allows one to define the modular automorphism group σ^Ω associated to the standard vector Ω by

$$\sigma_t^\Omega(x) := \Delta^{it}x\Delta^{-it}. \tag{3.3.1}$$

It shows in particular that $\sigma_t^\Omega(x) \in \mathfrak{M}$ for all $t \in \mathbb{R}$. Therefore, $(\mathfrak{M}, \sigma^\Omega)$ is a W^*-dynamical system. On entire analytic elements $x, y \in \mathfrak{M}_{\sigma^\Omega}$, it then holds

$$\begin{aligned} \omega(x^*\sigma_{-i}^\Omega(y)) &= \langle x\Omega, \Delta y \Delta^{-1}\Omega\rangle = \langle \Delta^{\frac{1}{2}}x\Omega, \Delta^{\frac{1}{2}}y\Omega\rangle = \langle Jx^*\Omega, Jy^*\Omega\rangle = \langle y^*\Omega, x^*\Omega\rangle \\ &= \omega(yx^*) \end{aligned}$$

for the state ω associated to Ω. Therefore, ω is a $(\sigma^\Omega, -1)$-KMS state on $\mathfrak{M}$. This is the final bridge between the theory of KMS states and Tomita-Takesaki theory.

If the $(\mathfrak{M}, \Omega)$ is the standard pair of a C^*-dynamical system $(\mathcal{A}, \alpha)$ with KMS state ω, then the relation between the modular automorphism group σ^Ω and the extended dynamics $\overline{\alpha}$ is the following.

Lemma 3.3.5. *Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$. Denote by $(\pi, \mathcal{H}, \Omega)$ the GNS triple and $\mathfrak{M}_\omega = \pi_\omega(\mathcal{A})''$.*

Then the dynamics $\overline{\alpha}$ (defined in Prop. 3.2.3) and the modular group σ^Ω (defined in Eq. (3.3.1)) satisfy the following relation

$$\overline{\alpha}_{-\beta t} = \sigma_t^\Omega.$$

Proof. Ω is a standard vector for $\mathfrak{M}_\omega$ by Proposition 3.2.3 and therefore defines the modular automorphism group σ^Ω. The associated state $\overline{\omega}$ is then a $(\sigma^\Omega, -1)$-KMS state as well as a $(\overline{\alpha}, \beta)$-KMS state.

The uniqueness assertion of the modular automorphism group in [BR97, Thm. 5.3.10] shows that $\overline{\alpha}$ equals σ^Ω up to rescaling. Following the proof of [BR97, Thm. 5.3.10], $e^{-\beta H}$ and Δ coincide on a common core. As both H and Δ are selfadjoint, it follows $e^{-\beta H} = \Delta$ as selfadjoint operators and thus

$$\sigma_t^\Omega(x) = \Delta^{it} x \Delta^{-it} = e^{i(-\beta t)H} x e^{-i(-\beta t)H} = \overline{\alpha}_{-\beta t}(x).$$

□

3.4 The Structure of the Set of KMS States

In this section, we turn to the study of the set of KMS states $\mathcal{S}_\beta(\mathcal{A}, \alpha)$ for a C^*-dynamical system $(\mathcal{A}, \alpha)$. It turns out that this set is weak-*-compact and convex. By the Krein-Milman theorem, it is the convex closure of its extreme points. These extreme points are then regarded as the pure thermodynamical phases of the system. The non-extremal KMS states are then regarded as mixtures of the pure phases. The extreme KMS states can be characterized as the factor KMS states of $\mathcal{A}$.

Before reviewing the structure theory of KMS states, we recall that the following notation. The *center* of a von Neumann algebra $\mathfrak{M}$ is defined by

$$\mathcal{Z}(\mathfrak{M}) := \{x \in \mathfrak{M} \mid xy = yx \, \forall y \in \mathfrak{M}\}$$

and the *centralizer* of $\mathfrak{M}$ w.r.t a state $\omega \in \mathcal{S}(\mathfrak{M})$ is similarly defined as

$$\mathcal{Z}_\omega(\mathfrak{M}) := \{x \in \mathfrak{M} \mid \omega(xy) = \omega(yx) \, \forall y \in \mathfrak{M}\}.$$

Clearly $\mathcal{Z}(\mathfrak{M}) \subset \mathcal{Z}_\omega(\mathfrak{M})$ for all states ω. Given finitely-twisted W^*-dynamical system $(\mathfrak{M}, \alpha, \gamma)$, the standard notation for the α- resp. γ-invariant elements is

$$\mathfrak{M}^\alpha := \{x \in \mathfrak{M} \mid \alpha_t(x) = x \, \forall t \in \mathbb{R}\} \quad \text{and} \quad \mathfrak{M}^\gamma := \{x \in \mathfrak{M} \mid \gamma_s(x) = x \, \forall s \in G\}.$$

Proposition 3.4.1. *Let $\mathfrak{M}$ be a concrete von Neumann algebra, Ω a cyclic unit vector, ω the corresponding state and α a weakly-continuous one-parameter group of *-automorphisms of $\mathfrak{M}$.*

If $\omega \in \mathcal{S}_\beta(\mathfrak{M}, \alpha)$, then

$$\mathcal{Z}_\omega(\mathfrak{M}) = \mathfrak{M}^\alpha \quad \textit{and} \quad \mathcal{Z}(\mathfrak{M}) \subset \mathfrak{M}^\alpha.$$

We refer to [BR97, Prop. 5.3.28] for the proof.

Proposition 3.4.2. *Let $\mathfrak{M}$ be a concrete von Neumann algebra, ω be a faithful normal state, σ the corresponding modular group and φ another normal state on $\mathfrak{M}$. The following are equivalent:*

1) *φ is a σ-KMS state;*

2) *There exists a positive operator T affiliated with $\mathcal{Z}(\mathfrak{M})$ s.t. $\varphi(x) = \omega(T^{\frac{1}{2}} x T^{\frac{1}{2}})$.*

If these statements are true, then T is unique.

In particular, ω is the unique σ-KMS state on $\mathfrak{M}$ if, and only if, $\mathfrak{M}$ is a factor.

Again, the proof can be found in [BR97, Prop. 5.3.29]. This proposition can now be used to prove the following theorem [BR97, Thm. 5.3.30] for a C^*-dynamical system.

Theorem 3.4.3. *Let $(\mathcal{A}, \alpha)$ be a unital C^*-dynamical system. Then:*

1) *$\mathcal{S}_\beta(\mathcal{A}, \alpha)$ is convex and weak-*-compact;*

2) *$\mathcal{S}_\beta(\mathcal{A}, \alpha)$ is a simplex;*

3) *$\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$ is an extreme point if, and only if, ω is a factor state.*

As $\mathcal{S}_\beta(\mathcal{A}, \alpha)$ is convex and weak-*-compact, the Krein-Milman Theorem can be applied and shows that $\mathcal{S}_\beta(\mathcal{A}, \alpha)$ is equal to the closed convex hull of its extreme points $\partial_e \mathcal{S}_\beta(\mathcal{A}, \alpha)$. Furthermore, if $\mathcal{S}_\beta(\mathcal{A}, \alpha)$ consists of a unique KMS state ω, then this state is automatically a factor state.

A similar statement holds for the set of γ-invariant KMS states $\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$.

Corollary 3.4.4. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted unital C^*-dynamical system. Then:*

1) *$\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ is convex and weak-*-compact;*

2) *$\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ is an extreme point if, and only if, $\mathcal{Z}(\mathfrak{M}_\omega)^{\overline{\gamma}} = \mathbb{C} \cdot \mathbb{1}$.*

Proof. 1): γ-invariance is a convex condition and moreover

$$\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma = \cap_{s \in G} \{\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha) \,|\, \omega \circ \gamma_s = \omega\}$$

is a closed subset of a compact set and thus compact. Here it was used that $\omega \mapsto \omega \circ \gamma_s$ is continuous in the weak-*-topology.

2): Following the proof of [BR97, Prop. 5.3.29] and [BR97, Thm. 5.3.30], every γ-invariant KMS state in the convex decomposition of ω is implemented by an operator $T \in \mathcal{Z}(\mathfrak{M}_\omega)^{\overline{\gamma}}$. Thus, if ω is an extreme point of $\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$, then this operator T is trivial and $\mathcal{Z}(\mathfrak{M}_\omega)^{\overline{\gamma}} = \mathbb{C} \cdot \mathbb{1}$ follows. Conversely, if ω is not extremal, then $\mathcal{Z}(\mathfrak{M}_\omega)^{\overline{\gamma}} \neq \mathbb{C} \cdot \mathbb{1}$. □

Analogous statements for W^*-dynamical systems $(\mathfrak{M}, \alpha)$ can be formulated, they however lack some important features. The normal states of a von Neumann algebra are weak-*-dense in the set of states. $\mathcal{S}_\beta(\mathfrak{M}, \alpha)$ is therefore not closed and compact in this topology and the Krein-Milman Theorem can not be applied.

Chapter 4

KMS States on Crossed Products

The focus of the previous chapters was on the introduction to the literature of crossed products and KMS states. We now turn to the original results of this thesis.

In this light, this chapter contains the study of KMS states on crossed products. We investigate in particular the question under which conditions a KMS state on $\mathcal{A}$ can be extended to a KMS state on the crossed product $\mathcal{A} \rtimes_\gamma G$. Moreover, we study the non-uniqueness of such extensions.

This amounts to the following physical question: Given an algebra of observables $\mathcal{A}$ with a time evolution α and a thermal equilibrium state ω at some temperature T. If we enlarge the algebra $\mathcal{A}$ by some (finite) symmetry group G, does the enlarged algebra $\mathcal{A} \rtimes G$ have a thermal equilibrium state? If yes, can we understand all the thermal equilibrium states of the enlarged algebra?

The structure of the crossed product allows for a canonical extension $\hat{\omega}^{\mathrm{can}}$ of a γ-invariant KMS state ω. The GNS representation of the canonical extension $\hat{\omega}^{\mathrm{can}}$ can then be studied in detail and allows a characterization of the space of extensions of the given KMS state ω in terms of either equivariant, positively compatible families of twisted KMS functionals or $\mathfrak{M}_\omega$-valued equivariant $\overline{\gamma}$-inner states on G. These characterizations allow us to structurally understand the set of all extensions of ω. We lastly shift the focus to the dynamics and discuss asymptotically abelian systems.

4.1 Canonical Extension of a KMS State

This section bridges the theory of KMS states and crossed products. We show that γ-invariant KMS states induce covariant representations of finitely-twisted C^*-dynamical system. The inclusion ι as well as the equivariant conditional expectation E allow us to relate the KMS states of $\mathcal{A}$ and $\mathcal{A} \rtimes_\gamma G$. We moreover introduce Gibbs representations as a means to construct multiple KMS states on $\mathcal{A} \rtimes_\gamma G$.

The following lemma is a generalization of Proposition 3.2.3 to finitely-twisted C^*-dynamical systems.

Lemma 4.1.1. *Let $(\mathcal{A}, \alpha, \gamma)$ be a C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$. Denote by $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ the GNS triple, by (J, Δ) the modular data and $\mathfrak{M}_\omega = \pi_\omega(\mathcal{A})''$.*

Then (π_ω, U, V) is a covariant representation of $(\mathcal{A}, \alpha, \gamma)$, where

$$U_t(\pi_\omega(a)\Omega) := \pi_\omega(\alpha_t(a))\Omega \quad \textit{and} \quad V_s(\pi_\omega(a)\Omega) := \pi_\omega(\gamma_s(a))\Omega,$$

and V commutes with the modular data J and Δ.

Moreover, $(\mathfrak{M}_\omega, \overline{\alpha}, \overline{\gamma})$ is a finitely-twisted W^*-dynamical system with standard vector Ω, where $\overline{\alpha}_t = \mathrm{Ad}_{U_t}$ and $\overline{\gamma}_s = \mathrm{Ad}_{V_s}$.

Proof. Proposition 3.2.3 already shows the assertions about U and $\overline{\alpha}$. As ω is γ-invariant, an analogous argument can be made and shows that V defines a unitary representation of G. As α and γ commute, it follows

$$V_s U_t(\pi_\omega(a)\Omega) = \pi_\omega(\gamma_s \circ \alpha_t(a))\Omega = \pi_\omega(\alpha_t \circ \gamma_s(a))\Omega = U_t V_s(\pi_\omega(a)\Omega)$$

for the unitary representations. As Ω is cyclic for $\pi_\omega(\mathcal{A})$, the representations U and V commute.

As γ is a *-action, the unitary representation V commutes with the Tomita operator

$$V_s S \pi_\omega(a)\Omega = V_s \pi_\omega(a^*)\Omega = \pi_\omega(\gamma_s(a)^*)\Omega = S V_s \pi_\omega(a)\Omega$$

on a core and thus with J and Δ. □

The following lemma shows that the pullback of a KMS state under a unital morphism of dynamical systems is a KMS state.

Lemma 4.1.2. *Let $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \sigma)$ be unital C^*-dynamical systems and $\iota : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism satisfying $\sigma_t \circ \iota = \iota \circ \alpha_t$ and $\iota(\mathbb{1}_\mathcal{A}) = \mathbb{1}_\mathcal{B}$.*

Let $\omega \in \mathcal{S}_\beta(\mathcal{B}, \sigma)$, then the pullback $\omega^\iota := \omega \circ \iota$ is an (α, β)-KMS state on $\mathcal{A}$.

Proof. ω^ι is a state on $\mathcal{A}$ since every *-morphism is continuous and $\omega^\iota(\mathbb{1}_A) = \omega(\mathbb{1}_B) = 1$. Take $a, b \in \mathcal{A}_\alpha$, then $\iota(a), \iota(b) \in \mathcal{B}_\sigma$ as ι is a morphism of C^*-dynamical systems.

$$\begin{aligned} \omega^\iota(a\alpha_{i\beta}(b)) &= \omega(\iota(a\alpha_{i\beta}(b))) = \omega(\iota(a)\iota(\alpha_{i\beta}(b))) = \omega(\iota(a)\sigma_{i\beta}(\iota(b))) \\ &= \omega(\iota(b)\iota(a)) = \omega(\iota(ba)) = \omega^\iota(ba) \end{aligned}$$

shows that the pullback ω^ι satisfies the KMS condition. □

We are interested in understanding the KMS states of the C^*-dynamical system $(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$. This will combine the results introduced in the preceding chapters. Recall that we have shown in Lemma 2.0.3 that the crossed product carries the dynamics

$$(\hat{\alpha}_t f)(s) = \alpha_t(f(s)), \quad f \in \mathcal{A} \rtimes_\gamma G,$$

and comes equipped with two natural maps, a faithful morphism of C^*-dynamical system

$$\iota : \mathcal{A} \hookrightarrow \mathcal{A} \rtimes G, \quad a \mapsto a \cdot \delta_e,$$

and an $\mathbb{R}$-equivariant faithful conditional expectation

$$E : \mathcal{A} \rtimes G \twoheadrightarrow \mathcal{A}, \quad f \mapsto f(e).$$

We now combine crossed products and KMS states with the following simple observation.

Proposition 4.1.3. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system. Then:*

1) *Any KMS state $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ of the crossed product pulls back to a γ-invariant KMS state $\omega := \hat{\omega} \circ \iota \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ of $\mathcal{A}$.*

2) *For every γ-invariant KMS state $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$, there exists a KMS state $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ such that $\hat{\omega} \circ \iota = \omega$.*

Proof. 1): By Lemma 2.0.3, ι is an inclusion of C^*-dynamical systems. This directly implies $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$ by Lemma 4.1.2. To see that ω is γ-invariant, note that

$$\begin{aligned}(\delta_g * f * \delta_{g^{-1}})(r) &= \sum_{s \in G} \delta_g(s) \gamma_s((f * \delta_{g^{-1}})(s^{-1} r)) = \sum_{s' \in G} \gamma_g(f(s')) \gamma_{gs'}(\delta_{g^{-1}}(s'^{-1} g^{-1} r)) \\ &= \gamma_g(f(g^{-1} r g))\end{aligned} \tag{4.1.1}$$

for $f \in \mathcal{A} \rtimes_\gamma G$. As δ_s is $\hat{\alpha}$-invariant, the KMS condition implies for $a \in \mathcal{A}$

$$\omega(\gamma_s(a)) = \hat{\omega}(\iota(\gamma_s(a))) = \hat{\omega}(\delta_s * \iota(a) * \delta_{s^{-1}}) \overset{\text{KMS}}{=} \hat{\omega}(\delta_{s^{-1}} * \delta_s * \iota(a)) = \omega(a).$$

2): The canonical faithful conditional expectation $E : \mathcal{A} \rtimes_\gamma G \to \mathcal{A}$ of Lemma 2.0.3 is $\mathbb{R}$-equivariant. Consider the state $\hat{\omega} := \omega \circ E$ on $\mathcal{A} \rtimes_\gamma G$, and entire elements $f, g : G \to \mathcal{A}_\alpha$. Using that γ and α commute and that ω is γ-invariant, we find

$$\begin{aligned}\hat{\omega}(f * \hat{\alpha}_{i\beta}(g)) &= \sum_{s \in G} \omega\Big(f(s) \gamma_s((\hat{\alpha}_{i\beta} g)(s^{-1}))\Big) = \omega\Big(f(s) \gamma_s(\alpha_{i\beta}(g(s^{-1})))\Big) \\ &\overset{\text{KMS}}{=} \sum_{s \in G} \omega(\gamma_s(g(s^{-1})) f(s)) = \sum_{s \in G} \omega(g(s^{-1}) \gamma_{s^{-1}}(f(s))) \\ &= \hat{\omega}(g * f)\end{aligned}$$

Thus $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$. ∎

We therefore see that understanding $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ amounts to understanding the extensions of γ-invariant KMS states $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ to $\hat{\alpha}$-KMS states on $\mathcal{A} \rtimes_\gamma G$. Note that a given KMS state $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$ can always be symmetrized to a γ-invariant KMS state via

$$\mathrm{Sym}_\gamma \, \omega := \frac{1}{|G|} \sum_{s \in G} \omega \circ \gamma_s, \quad \mathrm{Sym}_\gamma \, \omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma.$$

Therefore, every finitely-twisted C^*-dynamical system, which carries a KMS state allows for a γ-invariant KMS state. Note that this is a result of G being a finite and thus compact group.

As just shown, any γ-invariant $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$ has a *canonical extension*, denoted

$$\hat{\omega}^{\mathrm{can}} := \omega \circ E \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega, \tag{4.1.2}$$

where $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega$ denotes the KMS states $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ satisfying $\hat{\omega} \circ \iota = \omega$. In general, this extension is however not unique. We explain this non-uniqueness in an example below.

We call a covariant representation satisfying item 1) and 2) in the lemma below a *Gibbs representation of* $(\mathcal{A}, \alpha)$. Note that condition 1) is technically already part of Definition 2.0.2. As a further direct consequence of Lemma 4.1.2, every C^*-dynamical system having a Gibbs representation has a KMS state. Using Gibbs representations, we now give an example of a situation in which a KMS state has more than one extension to the crossed product. We want to emphasize that the Gibbs representations studied here are the core of quantum statistical mechanics.

Lemma 4.1.4. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system. If there exists a covariant representation (π, U, V) satisfying*

1) *π is non-degenerate for $\mathcal{A}$;*

2) *The selfadjoint generator H of U satisfies for some $\beta \neq 0$*

$$\mathrm{Tr}_{\mathcal{H}}(e^{-\beta H}) < \infty;$$

3) *There exists an $s \in G \backslash \{e\}$ satisfying*

$$\mathrm{Tr}_{\mathcal{H}}(e^{-\beta H} V_s) \neq 0. \tag{4.1.3}$$

Then the KMS state $\omega_\beta \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ defined by

$$\omega_\beta(a) := \frac{\mathrm{Tr}_{\mathcal{H}}(e^{-\beta H} \pi(a))}{\mathrm{Tr}_{\mathcal{H}}(e^{-\beta H})}$$

does not have a unique extension to the crossed product $\mathcal{A} \rtimes_\gamma G$.

Proof. It is well known that the trace class operator $e^{-\beta H}$ defines a KMS state g_β at inverse temperature β of the C^*-dynamical system $(\mathcal{B}(\mathcal{H}), \mathrm{Ad}_U)$ given by $g_\beta(T) = \frac{\mathrm{Tr}_\mathcal{H}(e^{-\beta H} T)}{\mathrm{Tr}_\mathcal{H}(e^{-\beta H})}$.

$(\pi \rtimes V, U)$ is a covariant representation of $(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ by Corollary 2.0.4 and satisfies $(\pi \rtimes V)(\delta_e) = \mathbb{1}$ by the non-degeneracy of π. Therefore Lemma 4.1.2 can be applied, showing that the pullback $\hat{\omega}_\beta^{\text{Gibbs}} := g_\beta \circ (\pi \rtimes V)$ is a KMS state on $\mathcal{A} \rtimes_\gamma G$. Similarly $\omega_\beta := \hat{\omega}_\beta^{\text{Gibbs}} \circ \iota \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$. It is moreover γ-invariant as (π, U, V) is a covariant representation and thus U and V commute. This shows that $\hat{\omega}_\beta^{\text{Gibbs}}$ is an extension of ω_β to the crossed product.

In view of condition (4.1.3), we have $\hat{\omega}_\beta^{\text{Gibbs}}(\delta_s) = g_\beta(V_s) \neq 0$ for some $s \neq e$. Since the canonical extension (4.1.2) satisfies $\hat{\omega}_\beta^{\text{can}}(\delta_s) = \omega_\beta(E(\delta_s)) = 0$, we have two different extensions $\hat{\omega}_\beta^{\text{can}}$ and $\hat{\omega}_\beta^{\text{Gibbs}}$ of ω_β. □

This lemma applies in particular to a concrete C^*-algebra $\mathcal{A}$ and a G-action γ s.t. $\mathcal{A} \rtimes G \simeq C^*(\mathcal{A}, U(G))$, see [SGL24] for a $\mathbb{Z}_2$-version.

On a side note, an interesting but complicated question is the following: Under which condition is the crossed product $\mathcal{A} \rtimes_\gamma G$ isomorphic to the C^*-algebra $C^*(\pi(\mathcal{A}), V(G))$ generated in a covariant representation (π, V). We discuss this for $G = \mathbb{Z}_2 = \{+1, -1\}$ as a preparation for Chapter 6 and refer to the literature of simple crossed product algebras for the general case [OP78; JL93].

Lemma 4.1.5. *Let $(\mathcal{A}, \gamma)$ be a simple unital $\mathbb{Z}_2$-twisted C^*-algebra and (π, V) be a covariant representation such that $V_{-1} \notin \pi(\mathcal{A})$.*

Then the C^*-algebra $C^*(\pi(\mathcal{A}), V(\mathbb{Z}_2))$ generated by $\pi(\mathcal{A})$ and V_{-1} is isomorphic to $\mathcal{A} \rtimes_\gamma \mathbb{Z}_2$.

Proof. Due to the simplicity of $\mathcal{A}$, we have $\mathcal{A} \simeq \pi(\mathcal{A})$ as C^*-algebras and moreover $\pi(\mathcal{A})$ is simple. The integrated representation $\hat{\pi} : \mathcal{A} \rtimes_\gamma \mathbb{Z}_2 \to C^*(\pi(\mathcal{A}), V(\mathbb{Z}_2))$, $\hat{\pi}(a, b) := \pi(a) + \pi(b)V_{-1}$ is a surjective homomorphism. To show that $\hat{\pi}$ is injective, note that $(a, b) \in \ker \pi$ implies $a \in \mathcal{I} := \{a \in \mathcal{A} : (\exists b \in \mathcal{A} : \pi(a) = \pi(b)V_{-1})\}$, which is a closed *-ideal in $\mathcal{A}$. By assumption, $V_{-1} \notin \pi(\mathcal{A})$, which implies $\mathbb{1} \notin \mathcal{I}$. Since $\mathcal{A}$ is simple, this yields $\mathcal{I} = \{0\}$; hence $\hat{\pi}$ is an isomorphism. □

4.2 GNS Representation of the Crossed Product

We begin this section with the study of the GNS representation of the canonical extension $\hat{\omega}^{\text{can}}$ of a γ-invariant KMS state ω. We then analyze the enveloping von Neumann algebra of $\hat{\pi}_\omega(\mathcal{A} \rtimes_\gamma G)$ and show that it is isomorphic to $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$.

The extensions of ω to $\mathcal{A} \rtimes_\gamma G$ can therefore be identified with a certain subset of $\mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)$.

The GNS representation of $\hat{\omega}^{\mathrm{can}}$ takes the following form.

Theorem 4.2.1. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$. Denote the corresponding covariant representation by (π_ω, U, V) with GNS space $\mathcal{H}_\omega$ and vector Ω. Then the following holds:*

1) *The GNS triple $(\hat{\pi}, \hat{\mathcal{H}}, \hat{\Omega})$ of $\hat{\omega}^{\mathrm{can}} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ is given by*

$$\hat{\mathcal{H}} = L^2(G, \mathcal{H}_\omega), \quad (\hat{\pi}(f)\varphi)(r) = \sum_{s\in G} \pi_\omega(f(s))V_s\varphi(s^{-1}r), \quad \text{and} \quad \hat{\Omega} = \Omega \cdot \delta_e;$$

2) *The modular group $\hat{U}$ and modular conjugation $\hat{J}$ of $\hat{\Omega}$ are given by*

$$(\hat{U}_t\varphi)(r) = U_t\varphi(r) \quad \text{and} \quad (\hat{J}\varphi)(s) = JV_s\varphi(s^{-1});$$

3) *The following map is an isomorphism of von Neumann algebras*

$$\overline{\pi} : \mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G \to \hat{\pi}(\mathcal{A} \rtimes_\gamma G)'', \quad (\overline{\pi}(f)\varphi)(r) := \sum f(s)V_s\varphi(s^{-1}r).$$

Proof. 1): We begin the proof by showing that $\hat{\pi}$ is a representation. The linearity follows by construction. To show the *-property, we first calculate the adjoint of $\hat{\pi}(f)$ by

$$\begin{aligned}\langle\varphi, \hat{\pi}(f)\psi\rangle &= \sum\langle\varphi(r), \pi_\omega(f(s))V_s\psi(s^{-1}r)\rangle \\ &= \sum\langle\varphi(s^{-1}r), \pi_\omega(f(s^{-1}))V_s^*\psi(r)\rangle \\ &= \sum\langle\pi_\omega(\gamma_s(f(s^{-1})^*))V_s\varphi(s^{-1}r), \psi(r)\rangle = \langle\hat{\pi}(f)^*\varphi, \psi\rangle,\end{aligned}$$

where the sum is taken over $s, r \in G$. The following calculation shows that $\hat{\pi}$ is compatible with the involution of $\mathcal{A} \rtimes_\gamma G$

$$\begin{aligned}(\hat{\pi}(f^*)\varphi)(r) &= \sum_{s\in G} \pi_\omega(f^*(s))V_s\varphi(s^{-1}r) = \sum_{s\in G} \pi_\omega(\gamma_s(f(s^{-1})^*))V_s\varphi(s^{-1}r) \\ &= (\hat{\pi}(f)^*\varphi)(r).\end{aligned}$$

Moreover, it is multiplicative by

$$\begin{aligned}(\hat{\pi}(f)\hat{\pi}(g)\varphi)(r) &= \sum \pi_\omega(f(s))V_s\pi_\omega(g(s'))V_{s'}\varphi(s'^{-1}s^{-1}r) \\ &= \sum \pi_\omega(f(s)\gamma_s(g(s')))V_{ss'}\varphi(s'^{-1}s^{-1}r) \\ &= \sum \pi_\omega(f(s)\gamma_s(g(s^{-1}\tilde{s})))V_{\tilde{s}}\varphi(\tilde{s}^{-1}r) \\ &= \sum \pi_\omega(f * g(\tilde{s}))V_{\tilde{s}}\varphi(\tilde{s}^{-1}r) = (\hat{\pi}(f * g)\varphi)(r),\end{aligned}$$

where the third equality is achieved by replacing s' by $\tilde{s} = ss'$. This shows that $\hat{\pi}$ is a representation of $\mathcal{A} \rtimes_\gamma G$. The vector $\hat{\Omega} \in L^2(G, \mathcal{H}_\omega)$ is clearly normalized and satisfies $(\hat{\pi}(f)\hat{\Omega})(s) = \pi_\omega(f(s))\Omega$ due to $\hat{\Omega}(s) = \Omega\delta_e(s)$ and $V_s\Omega = \Omega$. This relation readily implies

$$\begin{aligned}\langle\hat{\Omega}, \hat{\pi}(f)\hat{\Omega}\rangle &= \sum\langle\hat{\Omega}(s), (\hat{\pi}(f)\hat{\Omega})(s)\rangle = \sum \delta_e(s)\langle\Omega, \pi_\omega(f(s))\Omega\rangle \\ &= \langle\Omega, \pi_\omega(f(e))\Omega\rangle = \omega(f(e)) = \hat{\omega}^{\mathrm{can}}(f).\end{aligned}$$

The last line follows from Ω being the GNS vector of ω. As Ω is cyclic for $\pi_\omega(\mathcal{A})$, it follows that

$$\hat{\pi}(\mathcal{A} \rtimes_\gamma G)\hat{\Omega} = \mathrm{span}_{s\in G, a\in\mathcal{A}}\{\pi_\omega(a)\Omega \cdot \delta_s\}$$

is dense in $L^2(G, \mathcal{H}_\omega)$.

2): As the KMS state $\hat{\omega}^{\mathrm{can}}$ is $\hat{\alpha}$-invariant, the equation

$$\hat{U}_t\hat{\pi}(f)\hat{\Omega} = \hat{\pi}(\hat{\alpha}_t(f))\hat{\Omega}$$

defines the unique unitary $\mathbb{R}$-representation in the GNS space $(\hat{\pi}, \hat{\mathcal{H}}, \hat{\Omega})$ that implements $\hat{\alpha}$. This equation is satisfied by the above proposed representation $\hat{U}$ by the following calculation

$$\begin{aligned}(\hat{U}_t\hat{\pi}(f)\hat{\Omega})(r) &= U_t(\hat{\pi}(f)\hat{\Omega})(r) = U_t\pi_\omega(f(r))\Omega = \pi_\omega(\alpha_t(f(r)))\Omega \\ &= \pi_\omega((\hat{\alpha}_t f)(r))\Omega = (\hat{\pi}(\hat{\alpha}_t(f))\hat{\Omega})(r),\end{aligned}$$

where it was used that (π_ω, U, V) is a covariant representation.

We now show that the above proposed $\hat{J}$ is the corresponding modular conjugation. Take $f : G \to \mathcal{A}_\alpha$ and consider

$$\begin{aligned}(\hat{J}\hat{\Delta}^{\frac{1}{2}}\hat{\pi}(f)\hat{\Omega})(r) &= (\hat{J}\hat{\pi}(\hat{\alpha}_{\frac{-i}{2\beta}}(f))\hat{\Omega})(r) = JV_r(\hat{\pi}(\hat{\alpha}_{\frac{-i}{2\beta}}(f))\hat{\Omega})(r^{-1}) \\ &= JV_r\Delta^{\frac{1}{2}}\pi_\omega(f(r^{-1}))\Omega = \pi_\omega(\gamma_r(f(r^{-1})^*))\Omega = (\hat{S}\hat{\pi}(f)\hat{\Omega})(r).\end{aligned}$$

This shows that $\hat{J}$ is the anti-unitary involution in the polar decomposition of the Tomita operator $\hat{S}$ and hence it is the modular conjugation.

3): Consider the normal faithful extension $\overline{\omega}$ of ω to $\mathfrak{M}_\omega$. The composition with the normal faithful conditional expectation $E : \mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G \to \mathfrak{M}_\omega$ given in Lemma 2.0.5 defines a normal faithful state $\hat{\overline{\omega}}^{\mathrm{can}} := \overline{\omega} \circ E$. Consider the map

$$\overline{\pi} : \mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G \to \mathcal{B}(L^2(G, \mathcal{H}_\omega)), \quad (\overline{\pi}(f)\varphi)(r) := \sum f(s)V_s\varphi(s^{-1}r).$$

An analogous calculation as in 1) shows that $\overline{\pi}$ is a representation of $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ with cyclic vector $\hat{\Omega}$. Moreover,

$$\langle\hat{\Omega}, \overline{\pi}(f)\hat{\Omega}\rangle = \langle\Omega, f(e)\Omega\rangle = \overline{\omega}(f(e)) = \hat{\overline{\omega}}^{\mathrm{can}}(f).$$

Therefore, $\overline{\pi}$ is the GNS representation of the normal faithful state $\hat{\overline{\omega}}^{\text{can}}$ and thus normal by A.4 and moreover injective. The range of $\overline{\pi}$ contains $\hat{\pi}(\mathcal{A} \rtimes_\gamma G)$ by construction and thus also its enveloping von Neumann algebra $\hat{\pi}(\mathcal{A} \rtimes_\gamma G)''$. In order to show that $\hat{\pi}(\mathcal{A} \rtimes_\gamma G)''$ is the image of $\overline{\pi}$, consider $f \in \mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ and a sequence $(\pi_\omega(f_n))_{n\in\mathbb{N}}$ contained in $\pi_\omega(\mathcal{A}) \rtimes_\gamma G$ weakly converging to f. (This can be done in the regular representation of $\pi_\omega(\mathcal{A}) \rtimes_\gamma G$.) Then

$$\overline{\pi}(f) = \overline{\pi}(\text{WOT-lim } f_n) = \text{WOT-lim } \overline{\pi}(f_n) = \text{WOT-lim } \hat{\pi}(f_n) \in \hat{\pi}(\mathcal{A} \rtimes_\gamma G)''.$$

This proves the assertion. □

This theorem allows us to translate between the study of extensions of a KMS state ω to $\mathcal{A} \rtimes_\gamma G$ and the study of $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$. Here, Proposition 3.4.2 and Theorem 3.4.3 can be applied in the following way. Take $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ and consider $\hat{\omega}^{\text{can}} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G)$. Theorem 3.4.2 asserts that $\hat{\omega}^{\text{can}} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G)$ is an extreme point if, and only if, $\hat{\pi}(\mathcal{A} \rtimes_\gamma G)'' \simeq \mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ is a factor. If $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ is not a factor, then the non-uniqueness of $\hat{\omega}^{\text{can}}$ is characterized by $\mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)$ by Proposition 3.4.2.

Corollary 4.2.2. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$. Then the following holds:*

1) *$\hat{\omega}^{\text{can}}$ is an extreme point of $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ if, and only if, $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ is a factor;*

2) *If ω is not an extreme point of $\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$, then $\hat{\omega}^{\text{can}}$ is not an extreme point of $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$;*

Proof. 1): Follows directly from Theorem 3.4.3.

2): From Corollary 3.4.4 it follows that $\mathcal{Z}(\mathfrak{M}_\omega)^{\overline{\gamma}} \neq \mathbb{C} \cdot \mathbb{1}$, for ω not extremal in $\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$. Take a $\overline{\gamma}$-invariant central element z. Then $z \cdot \delta_e \in \mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)$ by

$$((z \cdot \delta_e) * f)(r) = zf(r) = f(r)z = f(r)\overline{\gamma}_r(z) = (f * (z \cdot \delta_e))(r).$$

As $z \cdot \delta_e \notin \mathbb{C} \cdot \mathbb{1}$, $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ is not a factor. □

The following corollary describes the elements p of $\mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)_+$, whose associated KMS states $\hat{\omega}_p$ pull back to ω on $\mathcal{A}$. Here we denote for a finitely-twisted von Neumann algebra $(\mathfrak{M}, \gamma)$

$$\mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)_{e,+} := \{f \in \mathcal{Z}(\mathfrak{M} \rtimes_\gamma G) \mid f \text{ positive} \quad \text{and} \quad f(e) = \mathbb{1}\}.$$

Corollary 4.2.3. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$.*

Every $p \in \mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)_{e,+}$ defines a KMS state $\hat{\omega}_p \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega$ by

$$\hat{\omega}_p(f) = \langle \hat{\Omega}, p^{\frac{1}{2}} \hat{\pi}(f) p^{\frac{1}{2}} \hat{\Omega} \rangle \quad \forall f \in \mathcal{A} \rtimes_\gamma G.$$

Proof. By Proposition 3.4.2, every positive operator $p \in \mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)$ defines a positive KMS functional ω_p on $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ and thus $\mathcal{A} \rtimes_\gamma G$. The assumption $p(e) = \mathbb{1}$ shows

$$\hat{\omega}_p(\iota(a)) = \langle \hat{\Omega}, p^{\frac{1}{2}} \iota(a) p^{\frac{1}{2}} \hat{\Omega} \rangle = \langle \hat{\Omega}, \iota(a) p \hat{\Omega} \rangle = \langle \Omega, \pi(a) p(e) \Omega \rangle = \omega(a),$$

which shows $\hat{\omega} \circ \iota = \omega$ and moreover implies that $\hat{\omega}_p$ is normalized. □

At this point, we do not show that every KMS state $\hat{\omega}$ on $\mathcal{A} \rtimes_\gamma G$ which pulls back to ω is of the proposed form. We postpone this discussion until Section 4.5, in which we analyze the positive central elements of $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ in more detail.

4.3 Twisted KMS Functionals

In Corollary 4.2.3 we have shown that every $p \in \mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)_{e,+}$ induces a KMS state on $\mathcal{A} \rtimes_\gamma G$, which pulls back to ω on $\mathcal{A}$. At this stage it is however not clear if $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ with $\hat{\omega} \circ \iota = \omega$ is given by a positive central element of $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$. Thus Proposition 3.4.2 is not directly applicable. In this section, we take a different approach of characterizing the KMS states of $\mathcal{A} \rtimes_\gamma G$ via twisted KMS functionals, which have been introduced in Definition 3.1.2. This will eventually allow us to show that the von Neumann algebra $\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G$ associated to $\hat{\omega}^{\text{can}}$ captures the extensions of ω.

The KMS states of the crossed product $\mathcal{A} \rtimes_\gamma G$ can be characterized in terms of KMS states and twisted KMS functionals of $\mathcal{A}$ in the following way.

Theorem 4.3.1. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system and let $(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ be its crossed product.*

There is a bijection

$$\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha}) \simeq \left\{ (\omega_s)_{s\in G} \,\middle|\, \begin{array}{ll} 1) & \omega_e \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma,\ \omega_s \in \mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma_s); \\ 2) & \omega_s \circ \gamma_r = \omega_{r^{-1}sr}; \\ 3) & [\omega_{rs^{-1}}(a_s^* a_r)]_{r,s\in G} \geq 0; \end{array} \right\}$$

carrying a KMS state $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ into the family $(\omega_s)_{s\in G}$ given by

$$\omega_s := \hat{\omega} \circ \iota_s, \quad \textit{where} \quad \iota_s(a) = a \cdot \delta_s,$$

and a family $(\omega_s)_{s\in G}$ into the KMS state

$$\hat{\omega}(f) := \sum_s \omega_s(f(s)). \tag{4.3.1}$$

Here property 3) *is understood as the positive semi-definiteness of the matrix* $[\omega_{rs^{-1}}(a_s^* a_r)]_{r,s\in G} \in M_{|G|}(\mathbb{C})$ *for all families* $(a_s)_{s\in G}$ *of elements in* $\mathcal{A}$.

A family $(\omega_s)_{s\in G}$ *of functionals on* $\mathcal{A}$ *is called a family of twisted KMS functionals if it satisfies property* 1)*, a* G*-equivariant family if it satisfies property* 2) *and positively compatible if it satisfies property* 3)*.*

We denote the set of G*-equivariant, positively compatible families of twisted KMS functionals by* $\mathcal{F}_\beta(\mathcal{A},\alpha,\gamma)^+$*. Given a KMS state* $\omega \in \mathcal{S}_\beta(\mathcal{A},\alpha)^\gamma$*, we denote the set of* G*-equivariant, positively compatible families of twisted KMS functionals s.t.* $\omega_e = \omega$ *by* $\mathcal{F}_\beta(\mathcal{A},\alpha,\gamma)^+_\omega$

Before giving the proof, we remark that the G-equivariance of a family of functionals $(\omega_s)_{s\in G}$ already implies the γ-invariance of ω_e. Furthermore, $\mathcal{F}_\beta(\mathcal{A},\alpha,\gamma)^+$ is a convex set and the family

$$\omega_s = \begin{cases} \omega & s = e; \\ 0 & s \neq e; \end{cases}$$

for a γ-invariant KMS state ω satisfies the conditions $1), 2)$ and 3). This is called the *trivial family of* ω and recovers the canonical extension $\hat{\omega}^{\text{can}}$.

Proof. Take a G-equivariant, positively compatible family $(\omega_s)_{s\in G}$ of twisted KMS functionals. Then $\hat{\omega}$ in equation (4.3.1) is clearly linear and $\hat{\omega}(\delta_e) = 1$. In order to show positivity of $\hat{\omega}$, let $f \in \mathcal{A} \rtimes_\gamma G$. Using the γ-invariance of ω_e and the equivariance, we find

$$\begin{aligned} \hat{\omega}(f^* * f) &= \sum_{s,r} \omega_s(\gamma_r(f(r^{-1})^* f(r^{-1}s))) = \sum_{\tilde{s},r} \omega_{r\tilde{s}}(\gamma_r(f(r^{-1})^* f(\tilde{s}))) \qquad (4.3.2) \\ &= \sum_{\tilde{s},r} \omega_{\tilde{s}r}(f(r^{-1})^* f(\tilde{s})) = \sum_{\tilde{s},r} \omega_{\tilde{s}r^{-1}}(f(r)^* f(\tilde{s})) \geq 0. \end{aligned}$$

Therefore $\hat{\omega}$ is a state on $\mathcal{A} \rtimes_\gamma G$ by property 3).

To verify the KMS condition, let $f, f' : G \to \mathcal{A}_\alpha$ and compute

$$\begin{aligned} \hat{\omega}(f' * \hat{\alpha}_{i\beta}(f)) &= \sum_{s,r} \omega_s(f'(r)\gamma_r((\hat{\alpha}_{i\beta} f)(r^{-1}s))) \qquad (4.3.3) \\ &= \sum_{s,r} \omega_s(f'(r)\alpha_{i\beta}(\gamma_r(f(r^{-1}s))) \\ &= \sum_{s,r} \omega_s(\gamma_r(f(r^{-1}s))\gamma_s(f'(r))) \\ &= \sum_{s,r} \omega_{r^{-1}sr}(f(r^{-1}s)\gamma_{r^{-1}s}(f'(r))) \\ &= \sum_{\tilde{s},r} \omega_{\tilde{s}r}(f(\tilde{s})\gamma_{\tilde{s}}(f'(r))) \\ &= \sum_{\tilde{s},\tilde{r}} \omega_{\tilde{r}}(f(\tilde{s})\gamma_{\tilde{s}}(f'(\tilde{s}^{-1}\tilde{r}))) \\ &= \hat{\omega}(f * f'), \end{aligned}$$

where the γ_s-twisted KMS condition of ω_s was used in the third equation and the G-equivariance was used in the fourth equation, followed by the changes of variables $\tilde{s} = r^{-1}s$ and $\tilde{r} = \tilde{s}r$. Thus, $\hat{\omega}$ is a KMS state for $\hat{\alpha}$.

Conversely, let $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ be a KMS state. By Lemma 4.1.3 1), its pullback $\omega \equiv \omega_e := \hat{\omega} \circ \iota_e$ is a γ-invariant KMS state for the dynamics α. Moreover, it is clear that $\omega_s := \hat{\omega} \circ \iota_s$ is a linear functional on $\mathcal{A}$. A look at calculation (4.3.3) with $f' = \iota_e(a)$ and $f = \iota_s(b)$ for $a, b \in \mathcal{A}_\alpha$ yields

$$\omega_s(a\alpha_{i\beta}(b)) = \omega_s(b\gamma_s(a))$$

as a consequence of the KMS property of $\hat{\omega}$. Thus ω_s is a γ_s-twisted KMS functional on $\mathcal{A}$.

It remains to show that the family $(\omega_s)_{s\in G}$ is equivariant and positively compatible. As δ_r is $\hat{\alpha}$-invariant it follows by the KMS condition and equation (4.1.1)

$$\omega_s(a) = \hat{\omega}(\iota_s(a)) \overset{\text{KMS}}{=} \hat{\omega}(\delta_r * \iota_s(a) * \delta_{r^{-1}}) = \omega_{rsr^{-1}}(\gamma_r(a)),$$

proving the equivariance. The positivity of $\hat{\omega}$, together with equivariance implies by equation (4.3.2) the positive compatibility of $(\omega_s)_{s\in G}$.

The operations relating $\hat{\omega}$ to $(\omega_s)_{s\in G}$ and vice versa are clearly inverses of each other, concluding the proof. □

In Lemma 4.3.2 we show that a inner group action γ by α-invariant unitaries induces an equivariant, positively compatible family of twisted KMS functionals. This can even be generalized to ω-weakly inner actions, which will be analyzed in Section 4.6. Moreover, the α-invariance of $\mathcal{V}_s$ can be dropped by going to the GNS representation, which will be shown in Lemma 4.4.6. This is one way of generating examples of such families of functionals.

If γ is not inner, then the idea of constructing such families is the following: Using the γ_s-twisted KMS condition 1), one shows whether or not a γ_s-twisted KMS functional ω_s exists or not. At this stage it might still be a unbounded. Assuming it exists, it will still depend on some parameters, e.g. the "phase" $\omega_s(\mathbb{1})$ is not fixed at this point. This is analogous to considering if the C^*-dynamical system carries a KMS state and then computing this KMS state. The G-equivariance 2) then imposes some relations between the parameters on which the ω_s depend. Namely, the ω_r for r in the conjugacy class of s are fixed. Proving or disproving the positive compatibility 3) of the thus constructed family of functionals is then the complicated part as this is a non-linear property.

We now turn to the simple case of inner twist γ.

Lemma 4.3.2. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely twisted C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$. Assume that γ is inner in $\mathcal{A}$ with unitary representation $\mathcal{V} : G \to \mathcal{A}$ (see Definition 4.4.1) satisfying $\mathcal{V}_s \in \mathcal{A}^\alpha$ for all $s \in G$.*

Then the family $(\omega_s)_{s\in G}$ defined by

$$\omega_s : \mathcal{A} \to \mathbb{C}, \quad \omega_s(a) := \omega(a\mathcal{V}_s)$$

is a G-equivariant, positively compatible family of twisted KMS functionals on $\mathcal{A}$.

Proof. These functionals are twisted KMS, since for $a, b \in \mathcal{A}_\alpha$

$$\omega_s(a\alpha_{i\beta}(b)) = \omega(a\alpha_{i\beta}(b)\mathcal{V}_s) = \omega(a\alpha_{i\beta}(b\mathcal{V}_s)) = \omega(b\mathcal{V}_s a) = \omega(b\gamma_s(a)\mathcal{V}_s) = \omega_s(b\gamma_s(a)),$$

where the α-invariance of $\mathcal{V}_s$ and the KMS condition was used. The family is equivariant by the γ-invariance of ω

$$\omega_s(\gamma_r^{-1}(a)) = \omega(\gamma_r^{-1}(a)\mathcal{V}_s) = \omega(\mathcal{V}_r^* a\mathcal{V}_r\mathcal{V}_s) = (\omega \circ \gamma_r^{-1})(a\mathcal{V}_{rsr^{-1}}) = \omega_{rsr^{-1}}(a).$$

The positive compatibility is a consequence of the positivity and γ-invariance of ω and the multiplicativity of $\mathcal{V}$. Take a family $(a_s)_{s\in G}$ of elements in $\mathcal{A}$ and a family of scalars $(\lambda_s)_{s\in G}$ and compute

$$\begin{aligned}\sum \overline{\lambda_s}\lambda_r\omega_{rs^{-1}}(a_s^* a_r) &= \sum \overline{\lambda_s}\lambda_r\omega(a_s^* a_r\mathcal{V}_{rs^{-1}}) = \sum \overline{\lambda_s}\lambda_r\omega(a_s^* a_r\mathcal{V}_r\mathcal{V}_s^*)\\ &= \sum \overline{\lambda_s}\lambda_r\omega(\mathcal{V}_s^* a_s^* a_r\mathcal{V}_r) = \omega((\sum_s \lambda_s a_s\mathcal{V}_s)^*(\sum_r \lambda_r a_r\mathcal{V}_r)) \geq 0.\end{aligned}$$

As this is the case for every pair of families $(a_s)_{s\in G}$ and $(\lambda_s)_{s\in G}$, the functionals $(\omega_s)_{s\in G}$ are positively compatible. □

Before studying the relation between the ω_s for different $s \in G$, recall that the adjoint ρ^* of a functional ρ on $\mathcal{A}$ is defined by

$$\rho^*(a) := \overline{\rho(a^*)}, \quad a \in \mathcal{A},$$

which is again a complex linear functional on $\mathcal{A}$. Now given a G-equivariant, positively compatible family of twisted KMS functionals $(\omega_s)_{s\in G}$, then the following relation for the inverse s^{-1} holds

$$\omega_{s^{-1}} = \omega_s^*,$$

which is a direct consequence of the selfadjointness of $[\omega_{rs^{-1}}(a_s^* a_r)]_{r,s\in G}$. Moreover, every ω_s is dominated by the KMS state $\omega \equiv \omega_e$ in the sense

$$|\omega_s(a^* b)|^2 \leq \omega(a^* a)\omega(b^* b),$$

which directly implies continuity of ω_s. These twisted KMS functionals are however typically not positive [BL99]. The domination property can be shown by fixing $\tilde{s} \in G$ and considering the matrix associated to the family

$$a_s = \begin{cases} a & s = e; \\ b & s = \tilde{s}; \\ 0 & s \neq e, \tilde{s}; \end{cases} .$$

In fact, for $G = \mathbb{Z}_2 = \{1, -1\}$ the condition ω_{-1} is γ_{-1}-twisted, hermitian and dominated by ω is necessary and sufficient for $(\omega_s)_{s\in\mathbb{Z}_2}$ to be a $\mathbb{Z}_2$-equivariant, positively compatible family of twisted KMS functionals, see [SGL24].

Lemma 4.3.3. *Let $\mathcal{A}$ be a C^*-algebra, $\omega \in \mathcal{S}(\mathcal{A})$ and ρ be a functional on $\mathcal{A}$. If ρ is hermitian, then the following are equivalent:*

1) *ρ is dominated by ω, i.e. $|\rho(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b)$;*

2) *$(\omega \pm \rho)$ are positive functionals (both combinations).*

Proof. 1) $\Rightarrow$ 2): First note that if ρ is dominated by ω, so is $-\rho$. Therefore, it is enough to show the positivity for one sign.

$$(\omega + \rho)(a^*a) = \omega(a^*a) + \rho(a^*a) \geq \omega(a^*a) - |\rho(a^*a)| \geq \omega(a^*a) - \omega(a^*a) = 0$$

2) $\Rightarrow$ 1): Consider the inequality for arbitrary $a, b \in \mathcal{A}$

$$(\omega \pm_1 \rho)((a \pm_2 b)^*(a \pm_2 b)) \geq 0.$$

Summing the terms $+_1+_2$ with $-_1-_2$ yields the inequality

$$0 \leq \omega(a^*a) + \omega(b^*b) + 2\operatorname{Re}\rho(a^*b).$$

By multiplying a with a phase, $\operatorname{Re}\rho(a^*b)$ can be replaced by $-|\rho(a^*b)|$. Changing $a \to a\sqrt{\omega(b^*b)}$ and $b \to b\sqrt{\omega(a^*a)}$ gives the desired inequality

$$2\sqrt{\omega(a^*a)\omega(b^*b)}|\rho(a^*b)| \leq 2\omega(a^*a)\omega(b^*b) \qquad \Longleftrightarrow \qquad |\rho(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b).$$

□

Note that for $\omega \pm \rho$ being positive, it is necessary that ρ is hermitian. A similar but weaker statement holds for arbitrary functionals ρ when considering the hermitian decomposition $\rho = \rho_+ + i\rho_-$, where

$$\rho_+ := \frac{1}{2}(\rho + \rho^*) \quad \text{and} \quad \rho_- := \frac{1}{2i}(\rho - \rho^*).$$

Corollary 4.3.4. *Let $\mathcal{A}$ be a C^*-algebra, $\omega \in \mathcal{S}(\mathcal{A})$ and ρ be a functional on $\mathcal{A}$. If there exist $\lambda_+, \lambda_- \in (1, \infty)$ s.t.*

$$\frac{1}{\lambda_+} + \frac{1}{\lambda_-} \leq 1, \quad (\omega \pm \lambda_+\rho_+) \geq 0 \quad \textit{and} \quad (\omega \pm \lambda_-\rho_-) \geq 0,$$

then ρ is dominated by ω.

Proof. From Lemma 4.3.3 it follows that $\lambda_+\rho_+$ and $\lambda_-\rho_-$ are dominated by ω. This directly implies

$$\begin{aligned}|\rho(a^*b)| &= |\rho_+(a^*b) + i\rho_-(a^*b)| \leq |\rho_+(a^*b)| + |\rho_-(a^*b)| \\ &\leq (\frac{1}{\lambda_+} + \frac{1}{\lambda_-})\sqrt{\omega(a^*a)\omega(b^*b)} \leq \sqrt{\omega(a^*a)\omega(b^*b)}.\end{aligned}$$

Squaring gives the sought after inequality. □

This small detour on dominated functionals was necessary for the upcoming discussion of the non-commutative Radon-Nikodým derivative. We now show that every functional ρ dominated by a state ω defines a non-commutative Radon-Nikodým derivative in the GNS representation of ω. If the functional is moreover γ-twisted KMS, then equation (4.3.5) connects the Radon-Nikodým derivative, the unitary implementing the twist and the modular conjugation.

Proposition 4.3.5. *Let $\mathcal{A}$ be a C^*-algebra, $\omega \in \mathcal{S}(\mathcal{A})$ with GNS triple $(\pi, \mathcal{H}, \Omega)$ and ρ a (hermitian) functional dominated by ω.*

Then there exists a unique operator $x'_\rho \in \pi(\mathcal{A})'$ s.t.

$$\rho(a) = \langle\Omega, \pi(a)x'_\rho\Omega\rangle, \tag{4.3.4}$$

which satisfies $\left\|x'_\rho\right\| \leq 1$ (and is selfadjoint). Conversely every $x' \in \pi(\mathcal{A})'$ satisfying $\|x'\| \leq 1$ defines a functional $\rho_{x'}$ on $\mathcal{A}$ dominated by ω.

Let furthermore $(\mathcal{A}, \alpha)$ be a C^*-dynamical system, $\gamma \in \mathrm{Aut}(\mathcal{A}), \gamma \circ \alpha = \alpha \circ \gamma$ and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ with modular data (J, Δ). Then ρ is a γ-twisted KMS functional if, and only if,

$$J(x'_\rho)^*J = V_\omega x'_\rho = x'_\rho V_\omega, \tag{4.3.5}$$

where V_ω is the unitary implementer of γ in the GNS space of ω.

Note that the Tomita-Takesaki Theorem allows for the rewriting of equations (4.3.4) and (4.3.5) as

$$\rho(a) = \langle\Omega, \pi(a)x_\rho\Omega\rangle \quad \text{and} \quad x_\rho = J(x_\rho)^*JV_\omega \tag{4.3.6}$$

in terms of $x_\rho := J(x'_\rho)^*J \in \mathfrak{M}_\omega$. The latter relation is a direct consequence of equation (4.3.5), whereas the former can be derived by considering

$$x'_\rho\Omega = Jx^*_\rho J\Omega = J(Jx_\rho V^*_\omega J)J\Omega = x_\rho\Omega.$$

Proof. Consider the densely defined sesquilinear form on $\mathcal{H}$ associated to ρ

$$h_\rho : \pi(\mathcal{A})\Omega \times \pi(\mathcal{A})\Omega \to \mathbb{C}, \quad (\pi(a)\Omega, \pi(b)\Omega) \mapsto \rho(a^*b)$$

and note that h_ρ is well-defined since ρ is dominated by ω. The sesquilinear form h_ρ is bounded by 1, since for all $a, b \in \mathcal{A}$,

$$|h_\rho(\pi(a)\Omega, \pi(b)\Omega)| = |\rho(a^*b)| \leq (\omega(a^*a)\omega(b^*b))^{\frac{1}{2}} = \|\pi(a)\Omega\|\|\pi(b)\Omega\|.$$

Therefore, h_ρ can be uniquely extended to a continuous sesquilinear form on $\mathcal{H}$ also denoted by h_ρ. By Lax-Milgram Theorem, there exists a unique operator $x'_\rho \in \mathcal{B}(\mathcal{H})$ satisfying

$$\rho(a^*b) = h_\rho(\pi(a)\Omega, \pi(b)\Omega) = \langle \pi(a)\Omega, x'_\rho \pi(b)\Omega \rangle, \quad \forall a, b \in \mathcal{A}.$$

It follows from the boundedness of h_ρ that $\left\| x'_\rho \right\| \leq 1$. Moreover, $x'_\rho \in \pi(\mathcal{A})'$ because

$$\langle \pi(a)\Omega, \left[x'_\rho, \pi(c) \right] \pi(b)\Omega \rangle = \rho(a^*cb) - \rho((c^*a)^*b) = 0.$$

The above reasoning can inverted. The functional $\rho_{x'}$ associated to $x' \in \pi(\mathcal{A})'$ with $\|x'\| \leq 1$ is dominated by ω. It is clear that ρ is hermitian if and only if the operator x'_ρ is selfadjoint.

Let now $(\mathcal{A}, \alpha)$ be a C^*-dynamical system and γ an automorphism commuting with the dynamics and ρ be a γ-twisted KMS functional. Following the proof of Lemma 4.1.1, there exists a unitary operator V_ω implementing γ and commuting with J. We rewrite, for $a, b \in \mathcal{A}_\alpha$, both sides of the twisted KMS condition.

$$\rho(a^*\alpha_{i\beta}(b)) = \rho(\alpha_{\frac{-i\beta}{2}}(a^*)\alpha_{\frac{i\beta}{2}}(b)) = \langle \Delta^{\frac{1}{2}}\pi(a)\Omega, x'_\rho \Delta^{\frac{1}{2}}\pi(b)\Omega \rangle$$
$$\rho(b\gamma(a^*)) = \langle \pi(b^*)\Omega, x'_\rho V_\omega \pi(a^*)\Omega \rangle = \langle \Delta^{\frac{1}{2}}\pi(a)\Omega, (V_\omega)^* J (x'_\rho)^* J \Delta^{\frac{1}{2}}\pi(b)\Omega \rangle$$

As $\Delta^{\frac{1}{2}}\pi(\mathcal{A}_\alpha)\Omega$ is dense in $\mathcal{H}$, it follows that $J(x'_\rho)^*J = V_\omega x'_\rho$. Clearly, this equation shows the converse direction as well. Note that, since ρ is γ-invariant by Lemma 3.1.3, x'_ρ commutes with V_ω. □

The above proposition is preparatory for one of the main theorems about KMS states on finitely-twisted systems. The statement however uses the γ-twisted center and von Neumann-valued states, which will be introduced in the next sections.

4.4 Twisted Center of a Von Neumann Algebra

This section starts with the introduction of the γ-twisted center $\mathcal{Z}(\mathfrak{M}, \gamma)$ of a von Neumann algebra $\mathfrak{M}$ with automorphism γ. It is exactly the set of operators $x \in \mathfrak{M}$ that commute with all $y \in \mathfrak{M}$ up to the twist γ. With the Kallman decomposition of the twisted center at hand, we show in Proposition 4.4.5 that for a standard pair $(\mathfrak{M}, \Omega)$, the twisted center $\mathcal{Z}(\mathfrak{M}, \gamma)$ consists exactly of the elements $x \in \mathfrak{M}$ satisfying $x = Jx^*JV$. Lastly, we discuss the relation between the twisted center and the modular group as well as the twisted center of different automorphisms.

We begin this section by defining innerness and outerness of a single automorphism and a group action.

Definition 4.4.1. *Let $(\mathcal{A}, \gamma)$ be a finitely-twisted C^*- or von Neumann algebra. We call γ inner in $\mathcal{A}$ if there exists a unitary representation $\mathcal{V} : G \to \mathcal{A}$ satisfying* $\mathrm{Ad}_{\mathcal{V}} = \gamma$ *on $\mathcal{A}$.*

Let $(\mathcal{A}, \gamma)$ be a finitely-twisted C^*-algebra and $\omega \in \mathcal{S}(\mathcal{A})^\gamma$. Then γ is called ω-weakly inner if $\overline{\gamma}$ is inner in $\mathfrak{M}_\omega$.

Note that if γ is inner in $\mathcal{A}$ and $\omega \in \mathcal{S}(\mathcal{A})^\gamma$, then $\overline{\gamma}$ is inner in $\mathfrak{M}_\omega$ by considering the unitary representation $\pi_\omega \circ \mathcal{V} : G \to \mathfrak{M}_\omega$.

Definition 4.4.2. *Let $\mathfrak{M}$ be a von Neumann algebra and $\gamma \in \mathrm{Aut}\,\mathfrak{M}$. The set*

$$\mathcal{Z}(\mathfrak{M}, \gamma) := \{x \in \mathfrak{M} \mid xy = \gamma(y)x \ \forall y \in \mathfrak{M}\}$$

is called the γ-twisted center of $\mathfrak{M}$. γ is called properly outer if $\mathcal{Z}(\mathfrak{M}, \gamma) = \{0\}$.

Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra. We call γ properly outer on $\mathfrak{M}$ if γ_s is properly outer for all $s \in G \backslash \{e\}$.

Note that the distinction into innerness and proper outerness is not a dichotomy. This already fails for a single automorphism and becomes even more complicated in the study of group actions. The notion of outerness of an automorphism in Definition 4.4.2 was introduced in [Kal69] under the term *freely acting*. However, we adapt the modern notation of *properly outer* in accordance with [Urs21], which also features a discussion of the different notions of innerness and outerness.

As a next step, we derive a different characterization of the twisted center of a von Neumann algebra $\mathfrak{M}$ with standard vector Ω in terms of the modular conjugation J. Here we focus on a single automorphism γ and postpone the discussion of a G-action until Proposition 4.4.8.

As a starting point, we introduce the Kallman decomposition of an automorphism $\gamma \in \mathrm{Aut}\,\mathfrak{M}$. For a γ-invariant central projection p, we denote $\mathfrak{M}_p = p\mathfrak{M}p$ and $\gamma_p = \gamma|_{\mathfrak{M}_p}$. In [Kal69, Thm. 1.11] it is shown by a Zorn argument that every *-automorphism of a von Neumann algebra can be uniquely decomposed into an inner and a properly outer part through a central projection.

Theorem 4.4.3. *Let $\mathfrak{M}$ be a von Neumann algebra and $\gamma \in \mathrm{Aut}\,\mathfrak{M}$.*

There exists a unique γ-invariant central projection $p \in \mathcal{Z}(\mathfrak{M})^\gamma$ s.t.

$$\mathfrak{M} = \mathfrak{M}_p \oplus \mathfrak{M}_{p^\perp} \quad \text{and} \quad \gamma = \gamma_p \oplus \gamma_{p^\perp},$$

where γ_p is inner in $\mathfrak{M}_p$ and $\gamma_{p^\perp}$ is properly outer on $\mathfrak{M}_{p^\perp}$.

Note that this decomposition of γ directly yields a decomposition of the γ-twisted center. The twisted center simply reduces to the center in case $\gamma = \mathbb{1}$,

$$\mathcal{Z}(\mathfrak{M}, \mathbb{1}) = \mathcal{Z}(\mathfrak{M})$$

and the automorphism $\mathbb{1}$ is always inner with unitary $v = \mathbb{1}$ and central projection $p = \mathbb{1}$.

Corollary 4.4.4. *Let $\mathfrak{M}$ be a von Neumann algebra and $\gamma \in \operatorname{Aut}\mathfrak{M}$.*

There exists a unique γ-invariant central projection $p \in \mathcal{Z}(\mathfrak{M})^\gamma$ and a (non-unique) partial isometry $v \in \mathcal{Z}(\mathfrak{M}_p, \gamma_p)$ satisfying $v^*v = p$ s.t.

$$\mathcal{Z}(\mathfrak{M}, \gamma) = \mathcal{Z}(\mathfrak{M}_p, \gamma_p) \oplus \mathcal{Z}(\mathfrak{M}_{p^\perp}, \gamma_{p^\perp}) = v \cdot \mathcal{Z}(\mathfrak{M}_p)^\gamma \oplus 0.$$

Proof. The Kallman decomposition yields the γ-invariant central projection p and induces the splitting

$$\mathcal{Z}(\mathfrak{M}, \gamma) = \mathcal{Z}(\mathfrak{M}_p, \gamma_p) \oplus \mathcal{Z}(\mathfrak{M}_{p^\perp}, \gamma_{p^\perp}).$$

$\mathcal{Z}(\mathfrak{M}_{p^\perp}, \gamma_{p^\perp}) = 0$ as $\gamma_{p^\perp}$ is properly outer. As γ_p is inner in $\mathfrak{M}_p$, there exists a partial isometry $v \in \mathfrak{M}$ implementing γ_p. This means concretely $vv^* = p$, $\operatorname{Ad}_v|_{\mathfrak{M}_p} = \gamma_p$ and $\gamma(v) = \gamma_p(v) = vvv^* = vp = v$. Then every $x \in \mathcal{Z}(\mathfrak{M}_p, \gamma_p)$ can be written as

$$x = px = v(v^*x) \in v \cdot \mathcal{Z}(\mathfrak{M}_p)^\gamma.$$

Here $v^*x \in \mathcal{Z}(\mathfrak{M}_p)$ due to the commutation relations induced by $v^* \in \mathcal{Z}(\mathfrak{M}_p, \gamma_p^{-1})$ and $x \in \mathcal{Z}(\mathfrak{M}_p, \gamma_p)$. The γ-invariance of v^*x can be seen by considering $\gamma(v^*x) = \gamma_p(v^*x) = vv^*xv^* = pxv^* = \gamma(v^*)x = v^*x$. Clearly, for every $z \in \mathcal{Z}(\mathfrak{M}_p)$ the product vz is contained in $\mathcal{Z}(\mathfrak{M}_p, \gamma_p)$. □

In the case that $\mathfrak{M}$ is a von Neumann algebra with standard vector Ω, the twisted center $\mathcal{Z}(\mathfrak{M}, \gamma)$ can be written in terms of the modular conjugation associated to Ω. For a finitely-twisted system, this allows for a connection between the modular data of Ω and the twisted center $\mathcal{Z}(\mathfrak{M}, \gamma_s)$.

Proposition 4.4.5. *Let $(\mathfrak{M}, \Omega)$ be a standard pair with modular data (J, Δ). Let $\gamma \in \operatorname{Aut}\mathfrak{M}$ s.t. the associated state ω is γ-invariant.*

Then

$$\mathcal{Z}(\mathfrak{M}, \gamma) = \{x \in \mathfrak{M} \mid x = Jx^*JV\},$$

where the operator implementing γ is the unique unitary extending

$$V : \mathfrak{M}\Omega \subset \mathcal{H} \to \mathfrak{M}\Omega \subset \mathcal{H}, \quad V(x\Omega) = \gamma(x)\Omega.$$

Proof. As ω is standard for $\mathfrak{M}$, V is well-defined. It is a densely-defined isometry with dense range by cylicity of Ω and γ-invariance of ω and thus its continuous extension is unitary. As γ is a *-automorphism, V commutes with the Tomita operator S on $\mathfrak{M}\Omega$. As this is a core for S, V commutes with the modular data.

Let $x \in \mathfrak{M}$ s.t. $Jx^*JV = x$. For $y \in \mathfrak{M}$ we find

$$xy = (Jx^*J)Vy = (Jx^*J)\operatorname{Ad}_V(y)V = (Jx^*J)\gamma(y)V = \gamma(y)Jx^*JV = \gamma(y)x.$$

As this is the defining relation of the twisted center, it shows $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$.

Let $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$. By restricting to the central projection p of Corollary 4.4.4, we can assume without loss of generality that $x = vz$, where v implements γ and $z \in \mathcal{Z}(\mathfrak{M})$. Then $v \in \mathfrak{M}^\gamma$ and furthermore $v \in \mathfrak{M}^\alpha = \mathcal{Z}_\Omega(\mathfrak{M})$ because

$$\langle \Omega, vy\Omega \rangle = \langle \Omega, v(yv)v^*\Omega \rangle = \langle \Omega, \gamma(yv)\Omega \rangle = \langle \Omega, yv\Omega \rangle, \qquad y \in \mathfrak{M}.$$

Considering the commutation relation implies $vV^* \in \mathfrak{M}'$. Using the modular conjugation J and the properties of Ω we find $Jv^*J\Omega = v\Omega = vV^*\Omega$ and with Ω separating for $\mathfrak{M}'$ that $Jv^*JV = v$. Focusing on x and using that $z \in \mathcal{Z}(\mathfrak{M})$ is equivalent to $z = Jz^*J$ shows

$$\begin{aligned} x = vz &= (Jv^*J)Vz = (Jv^*J)\,\mathrm{Ad}_v(z)V = (Jv^*J)zV \\ &= (Jv^*J)(Jz^*J)V = J(zv)^*JV = J(vz)^*JV = Jx^*JV, \end{aligned}$$

which proves the claim. □

We collect some properties of the γ-twisted center here, some which we have already used.

Lemma 4.4.6. *Let $\mathfrak{M}$ be a von Neumann algebra and $\gamma \in \mathrm{Aut}(\mathfrak{M})$.*

Then the following statements hold:

1) *$\mathcal{Z}(\mathfrak{M}, \gamma)$ is a WOT-closed complex subspace of $\mathfrak{M}$;*
2) *$x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ if, and only if, $x^* \in \mathcal{Z}(\mathfrak{M}, \gamma^{-1})$;*
3) *$\mathcal{Z}(\mathfrak{M}, \gamma) \subset \mathfrak{M}^\gamma$ and in particular $\mathcal{Z}(\mathfrak{M}, \gamma)$ is commutative and every $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ is normal;*
4) *For $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ with polar decomposition $x = u|x|$ it follows that $u \in \mathcal{Z}(\mathfrak{M}, \gamma)$ and $|x| \in \mathcal{Z}(\mathfrak{M})^\gamma$.*

If furthermore $(\mathfrak{M}, \Omega)$ is a standard pair with associated modular conjugation J and dynamics α s.t. the associated state ω is γ-invariant, then

$$\mathcal{Z}(\mathfrak{M}, \gamma) \subset \mathfrak{M}^\alpha.$$

Proof. 1): Clearly, $\mathcal{Z}(\mathfrak{M}, \gamma)$ is a complex subspace of $\mathfrak{M}$ and the defining relation depends WOT-continuously on $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$.

2): Consider $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ and $y \in \mathfrak{M}$. Then

$$x^*y = (y^*x)^* = (x\gamma^{-1}(y^*))^* = \gamma^{-1}(y)x^*$$

and thus $x \in \mathcal{Z}(\mathfrak{M}, \gamma^{-1})$.

3): This follows directly from the Kallman decomposition of Corollary 4.4.4. Every $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ can be written as $x = vz$ with a partial isometry v implementing γ_p

on $\mathfrak{M}_p$ and a central element $z \in \mathfrak{M}_p^\gamma$. Thus $\gamma(x) = \gamma_p(x) = v(vz)v^* = vz = x$. As a direct consequence, for $x, y \in \mathcal{Z}(\mathfrak{M}, \gamma)$ it follows $xy = \gamma(y)x = yx$ and $xx^* = \gamma(x)^* x = x^* x$.

4): Take $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ with polar decomposition $x = u|x|$. By item 3), x is normal. As $x^*x = xx^* \in \mathcal{Z}(\mathfrak{M})$ it follows $|x| = |x^*| \in \mathcal{Z}(\mathfrak{M})$. Therefore, both the domain projection $p = u^*u$ and the range projection $q = uu^*$ of x are central and coincide. Take $y \in \mathfrak{M}$ and $\varphi \in \ker x$ and compute

$$u\gamma^{-1}(y)\varphi = up\gamma^{-1}(y)\varphi = u\gamma^{-1}(y)p\varphi = 0 = yu\varphi.$$

Take $\varphi = |x|\psi \in \mathrm{Im}(|x|)$ which is dense in $\ker(x)^\perp$ as the domain and projections of x and $|x|$ are equal. Then

$$yu\varphi = yu|x|\psi = yx\psi = x\gamma^{-1}(y)\psi = u|x|\gamma^{-1}(y)\psi = u\gamma^{-1}(y)|x|\psi = u\gamma^{-1}(y)\varphi$$

and continuity imply $u \in \mathcal{Z}(\mathfrak{M}, \gamma)$.

Assume that $\mathfrak{M}$ has a standard vector Ω s.t. $\omega(\cdot \ddot{y}) = \langle \Omega, \cdot\, \Omega\rangle$ is γ-invariant and take $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$. Then $x \in \mathfrak{M}^\gamma$ by item 3) and furthermore $x \in \mathfrak{M}^\alpha = \mathcal{Z}_\Omega(\mathfrak{M})$ because

$$\langle \Omega, xy\Omega\rangle = \langle \Omega, \gamma(y)x\Omega\rangle = \langle \Omega, \gamma(yx)\Omega\rangle = \langle \Omega, yx\Omega\rangle.$$

□

After having introduced the twisted center, we now explain how it enters our discussion of KMS states on crossed products.

Assume that $(\mathfrak{M}, \Omega)$ is a standard pair and that ω is γ-invariant. Note that by Lemma 4.4.6 4), every nonzero $x \in \mathcal{Z}(\mathfrak{M}, \gamma)$ gives rise to a γ-invariant KMS state ν_x on $\mathfrak{M}$ by

$$\nu_x(y) := \left\| |x|^{\frac{1}{2}}\Omega \right\|^{-2} \langle |x|^{\frac{1}{2}}\Omega, y|x|^{\frac{1}{2}}\Omega\rangle,$$

due to the correspondence between normal KMS states on $\mathfrak{M}$ and the operators affiliated to $\mathcal{Z}(\mathfrak{M})$, see [BR97, Proposition 5.3.29]. This coincides with the abstract polar decomposition of twisted KMS functionals. Every nonzero twisted KMS functional ρ defines a KMS state $|\rho|$ by polar decomposition of functionals and subsequent normalization, the resulting twist $\overline{\gamma}$ in the GNS space of $|\rho|$ is then given by an inner unitary, see [BL99].

In the interest of a criterion that can be checked in concrete applications, we also note the following more concrete situation.

Proposition 4.4.7. *Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system, $\gamma \in \mathrm{Aut}(\mathcal{A}), \gamma \circ \alpha = \alpha \circ \gamma$ and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ with associated standard pair $(\mathfrak{M}, \Omega)$. Suppose $\mathcal{A}$ contains a sequence $(v_n)_{n\in\mathbb{N}}$ of unitary elements that satisfy*

$$\lim_{n,m\to\infty} \omega(v_n^* v_m) = 1, \qquad \lim_{n\to\infty} \|v_n a v_n^* - \gamma(a)\| = 0, \qquad a \in \mathcal{A}_0,$$

where $\mathcal{A}_0 \subset \mathcal{A}_\alpha$ is a norm dense *-subalgebra invariant under the dynamics. Then the limit $\mathcal{V} = \text{SOT-lim}\,\pi(v_n) \in \mathcal{Z}(\mathfrak{M}_\omega, \overline{\gamma})$ exists, is unitary and

$$\rho(a) := \lim_{n\to\infty} \omega(av_n) = \langle \Omega, \pi(a)\mathcal{V}\Omega\rangle, \qquad a \in \mathcal{A},$$

is a γ-twisted KMS functional dominated by ω, and the family of all such functionals is $\{\rho(\,\cdot\, z) \mid z \in \mathcal{Z}(\mathfrak{M}),\ \|z\| \leq 1\}$.

Proof. We first show the existence of the limit defining ρ. For $a \in \mathcal{A}$, we estimate

$$|\omega(a(v_n - v_m))|^2 \leq \omega(a^*a)\omega(2 - v_n^* v_m - v_m^* v_n) = 2\omega(a^*a)\cdot(1 - \operatorname{Re}\omega(v_n^* v_m)),$$

where we have used $\omega = \omega^*$ in the last step. As $\omega(v_n^* v_m) \to 1$, this shows that ρ exists as a functional on $\mathcal{A}$.

Similarly, for $a \in \mathcal{A}, b \in \mathcal{A}_0$, we have

$$\omega(av_n b) = \omega(a[v_n b v_n^* - \gamma(b)]v_n) + \omega(a\gamma(b)v_n) \to \rho(a\gamma(b))$$

because $\|v_n b v_n^* - \gamma(b)\| \to 0$. By uniform boundedness, this implies the existence of the WOT-limit $\mathcal{V} := \lim_n \pi(v_n)$. But as $\|\mathcal{V}\Omega\|^2 = \lim_{n,m}\omega(v_n^* v_m) = 1$, we also have $\|\pi(v_n)\Omega - \mathcal{V}\Omega\|^2 \to 1 - \|\mathcal{V}\Omega\|^2 = 0$ and hence $\|(\pi(v_n) - \mathcal{V})x'\Omega\| \to 0$ for all $x' \in \mathfrak{M}'$. As Ω is cyclic for $\mathfrak{M}'$, we arrive at $\pi(v_n) \to \mathcal{V}$ in SOT, and hence at the unitarity of $\mathcal{V}$. It is then clear from our assumptions that $\mathcal{V}x\mathcal{V}^* = \overline{\gamma}(x)$ for all $x \in \mathfrak{M}$. It now follows from Proposition 4.4.5 that $\mathcal{V} = J\mathcal{V}^*JV$, where V is the unitary operator associated to $\overline{\gamma}$ introduced in Proposition 4.4.5. This relation is now the defining relation of the non-commutative Radon-Nikodým derivative of Proposition 4.3.5, which proves the claim. □

We now turn to the discussion of the twisted centers given by a finite group action. As Proposition 4.4.8 shows, the Kallman decomposition respects the inversion of G as well as its conjugacy classes. The multiplicative structure of the group G and the Kallman decomposition do however not match in general.

Proposition 4.4.8. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra and consider for every $s \in G$ the Kallman decomposition $\mathcal{Z}(\mathfrak{M}, \gamma_s) = v_s \cdot \mathcal{Z}(\mathfrak{M}_{p_s}) \oplus 0$.*

Then following holds:

1) *$\mathcal{Z}(\mathfrak{M}, \gamma_e) = \mathcal{Z}(\mathfrak{M})$, $v_e = \mathbb{1}$ and $p_e = \mathbb{1}$;*

2) *$\mathcal{Z}(\mathfrak{M}, \gamma_{s^{-1}}) = v_s^* \cdot \mathcal{Z}(\mathfrak{M}_{p_s}) \oplus 0$, in particular $v_{s^{-1}} = v_s^*$ and $p_{s^{-1}} = p_s$;*

3) *$\mathcal{Z}(\mathfrak{M}, \gamma_{rsr^{-1}}) = \gamma_r(v_s) \cdot \mathcal{Z}(\mathfrak{M}_{\gamma_r(p_s)}) \oplus 0$, in particular $v_{rsr^{-1}} = \gamma_r(v_s)$ and $p_{rsr^{-1}} = \gamma_r(p_s)$;*

4) *$\mathcal{Z}(\mathfrak{M}, \gamma_s) \cdot \mathcal{Z}(\mathfrak{M}, \gamma_r) \subset \mathcal{Z}(\mathfrak{M}, \gamma_{sr})$, in particular $p_s p_r \leq p_{sr}$.*

Note that p_s is the unique γ_s-invariant central projection s.t. $\gamma_s|_{p_s}$ is inner and $\gamma_s|_{p_s^\perp}$ is properly outer. The partial isometry v_s is however only unique up to central elements, i.e. one can choose $v_{s^{-1}} = v_s^*$ and $v_{rsr^{-1}} = \gamma_r(v_s)$. In case $\mathfrak{M}$ is a factor, the partial isometries are either 0 or unitary.

For an abelian group G, all the Kallman projections and partial isometries are γ-invariant as the conjugacy classes consist of only a single element.

Proof. 1): Is clear since $\gamma_e = \mathbb{1}$.

2): Take the Kallman decomposition $\mathcal{Z}(\mathfrak{M}, \gamma_{s^{-1}}) = v_{s^{-1}} \cdot \mathcal{Z}(\mathfrak{M}_{p_{s^{-1}}})$. Then $v_{s^{-1}}^* \in \mathcal{Z}(\mathfrak{M}, \gamma_s)$ and thus $p_s \geq v_{s^{-1}}^* v_{s^{-1}} = p_{s^{-1}}$. Using this inequality twice shows $p_s = p_{s^{-1}}$ and $v_{s^{-1}}$ can be chosen as v_s^*.

3): Holds by an argument analogous to 2) by replacing inversion with conjugation and realizing $p_{rsr^{-1}} = \gamma_r(v_s^*)\gamma_r(v_s) = \gamma_r(p_s)$.

4): Clearly $v_s v_r \in \mathcal{Z}(\mathfrak{M}, \gamma_{sr})$ and thus $p_{sr} \geq (v_s v_r)^* v_s v_r = p_s p_r$. □

The following is a simple example of a finite group action where item 4) is a proper inequality. Take a von Neumann algebra $\mathcal{N}$ and a properly outer $\mathbb{Z}_2$-automorphism γ. Then define $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and take

$$\mathfrak{M} = \mathcal{N} \oplus \mathcal{N} \oplus \mathcal{N}, \quad \tilde{\gamma}(s, r) = \gamma^s \oplus \gamma^{s+r} \oplus \gamma^r.$$

Considering $\mathbb{Z}_2 = \{0, 1\}$ here clearly makes $\overline{\gamma}$ a G-action on $\mathfrak{M}$. The Kallman projections satisfy $p_{(1,0)} = 0 \oplus 0 \oplus \mathbb{1}$, $p_{(0,1)} = \mathbb{1} \oplus 0 \oplus 0$ and $p_{(1,1)} = 0 \oplus \mathbb{1} \oplus 0 > p_{(1,0)} p_{(0,1)}$.

Considering that the Kallman decomposition respects the inversion and conjugation of the group G, we find the following. If $p_s \neq 0$ for some $s \in G \backslash \{e\}$, then $p_r \neq 0$ for all r in the normal subgroup generated by s. This idea is has the following consequence.

Corollary 4.4.9. *Let* $(\mathfrak{M}, \gamma)$ *be a finitely-twisted von Neumann algebra and* G *be simple.*

Then one of the following is true:

1) $p_s \neq 0$ *for all* $s \in G$ *or;*

2) $p_s = 0$ *for all* $s \in G \backslash \{e\}$.

If $\mathfrak{M}$ *is moreover a factor, then item* 1) *is replaced by* $p_s = \mathbb{1}$ *for all* $s \in G$.

Proof. Assume there exists $s \in G \backslash \{e\}$ s.t. $p_s \neq 0$. Then $p_{s^{-1}} = p_s$ and $p_{s^n} \geq (p_s)^n = p_s$ for $n \in \mathbb{Z}$ by Proposition 4.4.8. Thus $p_r \neq 0$ for elements of the subgroup G_s generated by s. We similarly have $p_{rs^n r-1} = \gamma_r(p_{s^n}) \geq \gamma_r(p_s) \neq 0$. The normal subgroup N_s generated by s is exactly given such elements $rs^n r^{-1}$ for some $r \in G$ and $n \in \mathbb{Z}$. Therefore, $p_{\tilde{r}} \neq 0$ for all $\tilde{r} \in N_s$. As G is simple, we have $N_s = G$.

The converse of the above assumption is exactly item 2).

If $\mathfrak{M}$ is a factor, then $p_s = \mathbb{1}$ or $p_s = 0$. □

Corollary 4.4.10. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra, where $\mathfrak{M}$ is a factor and G is simple and abelian.*

Then γ is either inner or properly outer in $\mathfrak{M}$.

Proof. By Corollary 4.4.9, either all projections are $\mathbb{1}$ or vanish for $s \neq e$. If all projections vanish, then γ is properly outer. Thus, assume that $p_s = \mathbb{1}$ for all $s \in G$. Take a fixed element $s_0 \neq e$ with unitary v_{s_0}. As G is finite, s_0 is of finite order N. As $v_{s_0}^N \in \mathcal{Z}(\mathfrak{M}) = \mathbb{C} \cdot \mathbb{1}$, we can take the N-th root and assume $v_{s_0}^N = \mathbb{1}$. This allows us to construct a unitary representation of G_{s_0} by

$$\mathcal{V} : G_{s_0} \to \mathfrak{M}, \quad \mathcal{V}(s_0^n) := v_{s_0}^n, \quad \forall n \in \mathbb{Z}.$$

As G is abelian and simple, it follows that this subgroup is already $G_{s_0} = N_{s_0} = G$. Therefore, $\gamma = \mathrm{Ad}_{\mathcal{V}}$ is an inner action. □

Similar arguments can be done by assuming that G has a generating set S s.t. each $p_s \neq 0$ for $s \in S$. We will however not explore the finer relation between the Kallman decomposition and the group structure here.

4.5 $\mathfrak{M}$-Valued Equivariant γ-Inner States on Finite Groups

Operator-valued states on finite groups are introduced and combined with the notion of the twisted center discussed before. We then proceed to show that these equivariant γ-inner states $\mathcal{S}_\gamma(G, \mathfrak{M})$ lie in the center of $\mathfrak{M} \rtimes_\gamma G$ and characterize the extensions of ω to $\mathcal{A} \rtimes_\gamma G$ when applied to a $\mathfrak{M} = \mathfrak{M}_\omega$.

As a preparation, we make the following definition. This definition is equivalent to the usual definition of operator-valued states on (topological) groups as is explained below.

Definition 4.5.1. *Let $\mathfrak{M}$ be a von Neumann algebra and G be a finite group. A function $\varphi : G \to \mathfrak{M}$ is called $\mathfrak{M}$-valued state of G if $\varphi(e) = \mathbb{1}$ and the matrix $[\varphi] \in M_{|G|}(\mathfrak{M})$ defined by $[\varphi]_{rs} := \varphi(rs^{-1})$ is positive-semidefinite. We denote the set of $\mathfrak{M}$-valued states of G by $\mathcal{S}(G, \mathfrak{M})$.*

The $\mathfrak{M}$-valued state defined by

$$\varphi_{\mathrm{triv}} : G \to \mathfrak{M}, \quad \varphi(e) = \mathbb{1}, \varphi(s) = 0 \,\forall\, s \neq e$$

is called the trivial $\mathfrak{M}$*-valued state of* G. *The set of states* $\mathcal{S}(G, \mathfrak{M})$ *is called trivial if*

$$\mathcal{S}(G, \mathfrak{M}) = \{\varphi_{\text{triv}}\}.$$

Note that $[\varphi]$ a priori depends on an enumeration $G = \{s_1, \ldots, s_{|G|}\}$. The positive-semidefiniteness of $[\varphi]$ is however invariant under changes of enumeration, as such a change is implemented by conjugation with a unitary matrix $U \in \mathfrak{M}_{|G|}(\mathbb{C})$.

The above definition of $\mathfrak{M}$-valued states of a finite group G coincides with the usual definition of operator-valued states on involutive semigroups when passing to a representation, which we show in the following. See [Nee11, Chap. 3.1] for an in depth introduction to positive definite functions on involutive semigroups. As a passing remark, we mention that there is a GNS-type theorem for positive definite functions as well.

As a preliminary, we introduce the *inflation of* **-morphisms*. Take a representation $(\pi, \mathcal{H})$ of $\mathfrak{M}$. Then its inflation, see [Mur90, Sec. 3.4],

$$\pi_n : M_n(\mathfrak{M}) \to M_n(\pi(\mathfrak{M})), \quad [x_{ij}] \mapsto [\pi(x_{ij})]$$

is again a *-morphism and thus π is completely positive. In particular, $\pi_{|G|}([\varphi])$ is positive-semidefinite.

Assume that φ is an operator-valued state in the sense that $\varphi(e) = \mathbb{1}$ and for every representation $(\pi, \mathcal{H})$ and finite sequence $(r_1, v_1), \ldots, (r_n, v_n)$ in $G \times \mathcal{H}$, the following inequality holds

$$\sum_{i,j=1}^{n} \langle v_i, \pi(\varphi(r_i r_j^{-1}))v_j \rangle \geq 0.$$

We then label the elements of G, i.e. $G = \{r_1, \ldots r_{|G|}\}$ and choose an arbitrary sequence $v_1, \ldots, v_{|G|}$ in $\mathcal{H}$ and define $v = (v_1, \ldots, v_{|G|}) \in \oplus_{i=1}^{|G|} \mathcal{H}$. Then

$$\langle v, \pi_{|G|}([\varphi])v \rangle = \sum_{i,j=1}^{|G|} \langle v_i, \pi(\varphi(r_i r_j^{-1}))v_j \rangle \geq 0$$

by assumption. Thus $\pi_{|G|}([\varphi])$ is positive-semidefinite for every representation π and thus $[\varphi]$ is positive-semidefinite.

Assume now that φ is an $\mathfrak{M}$-valued state in the sense of Definition 4.5.1 and label the elements of G. Take a finite sequence $(r_1, v_1), \ldots, (r_n, v_n)$ in $G \times \mathcal{H}$. We inflate the sequence of vectors to a sequence $w_1, \ldots, w_n$ of vectors in $\mathcal{H}^{|G|}$ by choosing $w_i = (0, \ldots, v_i, \ldots 0)$, where v_i is at the r_i-th position. This allows us to rewrite

$$\sum_{i,j=1}^{n} \langle v_i, \pi(\varphi(r_i r_j^{-1}))v_j \rangle = \sum_{i,j=1}^{n} \langle w_i, \pi_{|G|}([\varphi])w_j \rangle \geq 0.$$

Thus, φ is positive-semidefinite in the sense of a positive definite operator-valued function on G.

Having shown that our definition of operator-valued state coincides with the literature, we state some elementary properties of $\mathfrak{M}$-valued states.

Lemma 4.5.2. *Let $\mathfrak{M}$ be von Neumann algebras and G a finite group. Then the following holds:*

1) *For $\mathfrak{M} = \mathbb{C}$, one recovers the usual states $\mathcal{S}(G)$ on (finite) groups;*

2) *If $\varphi \in \mathcal{S}(G, \mathfrak{M})$, then $\varphi(r^{-1}) = \varphi(r)^*$ and $\|\varphi(r)\| \leq 1$ for all $r \in G$;*

3) *$\mathcal{S}(G, \mathfrak{M})$ is a convex set and weakly closed.*

Proof. 1): This is clear from the above discussion.

2) : As the matrix $[\varphi] \in M_{|G|}(\mathfrak{M})$ is positive-semidefinite, it is selfadjoint. It directly follows

$$\varphi(r) = [\varphi]_{re} = ([\varphi]^*)_{re} = ([\varphi]_{er})^* = \varphi(r^{-1})^*.$$

Going over to a faithful representation of $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$ and choosing $w = (v_e, 0, \ldots, 0, v_r, 0, \ldots, 0)$, where v_r is at the r-th position, yields

$$\begin{pmatrix} \|v_e\|^2 & \langle v_e, \varphi(r)^* v_r \rangle \\ \langle v_r, \varphi(r) v_e \rangle & \|v_r\|^2 \end{pmatrix} \geq 0.$$

By positivity of the determinant, it follows $\|\varphi(r)\| \leq 1$.

3): Clearly, $\mathcal{S}(G, \mathfrak{M})$ is a convex set. Moreover, if $\varphi(r) = \text{WOT-lim}\, \varphi_i(r)$, then for $w \in \mathcal{H}^{|G|}$

$$\varphi(e) = \text{WOT-lim}\, \varphi_i(e) = \text{WOT-lim}\, \mathbb{1} = \mathbb{1} \quad \text{and} \quad \langle w, [\varphi] w \rangle = \lim \langle w, [\varphi_i] w \rangle \geq 0.$$

□

This notion of positivity introduced in Definition 4.5.1 is similar to the positive compatibility 3) of Theorem 4.3.1. We now introduce the notion of G-equivariant γ-inner $\mathfrak{M}$-valued states in accordance with the G-equivariance and twisted KMS property of the family of functionals discussed in Theorem 4.3.1.

The G-equivariant γ-inner $\mathfrak{M}$-valued states describe the intricate interplay between positivity and innerness of γ.

Definition 4.5.3. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra. We denote the set of G-equivariant γ-inner $\mathfrak{M}$-valued states by*

$$\mathcal{S}_\gamma(G, \mathfrak{M}) := \{\varphi \in \mathcal{S}(G, \mathfrak{M}) \mid \varphi(r) \in \mathcal{Z}(\mathfrak{M}, \gamma_r) \quad \text{and} \quad \gamma_s(\varphi(r)) = \varphi(srs^{-1})\ \forall s \in G\}.$$

Note that, independent of the group structure of G and the action γ, the state φ_{triv} is always equivariant and γ-inner.

For the simple case $\mathfrak{M} = \mathbb{C}$, every unital *-automorphism γ_r is trivial and thus

$$\mathcal{S}_\gamma(G, \mathbb{C}) = \{\varphi \in \mathcal{S}(G) \mid \varphi(r) = \varphi(srs^{-1}) \ \forall s \in G\} = \mathcal{S}_c(G)$$

coincides with the set of states which are constant on conjugacy classes. This gives a connection to the well-known theory of characters of finite groups, which will be explored in Section 4.6.

The following lemma shows that every inner group action γ induces an G-equivariant γ-inner $\mathfrak{M}$-valued state. This can be thought of as the analogue of Lemma 4.3.2 for $\mathfrak{M}$-valued states. Moreover, this result says that one should think of a G-equivariant γ-inner $\mathfrak{M}$-valued state as a kind of "partial unitary representation". This idea can be made precise by a GNS-type theorem [Nee11].

Lemma 4.5.4. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra and γ inner in $\mathfrak{M}$ via the unitary representation $\mathcal{V}$.*

Then

$$\mathcal{V} \in \mathcal{S}_\gamma(G, \mathfrak{M}).$$

Proof. $\mathcal{V}$ is clearly equivariant by $\gamma_r(\mathcal{V}(s)) = \mathcal{V}(r)\mathcal{V}(s)\mathcal{V}(r)^* = \mathcal{V}(rsr^{-1})$ and γ-inner by definition. Passing to a representation and taking $(w_1, \ldots, w_{|G|})$ in $\mathcal{H}$, the positivity follows from

$$\sum_{r,s} \langle w_r, [\mathcal{V}]_{rs} w_s \rangle = \sum_{r,s} \langle w_r, \mathcal{V}(rs^{-1}) w_s \rangle = \sum_{r,s} \langle \mathcal{V}(r)^* w_r, \mathcal{V}(s)^* w_s \rangle \geq 0.$$

□

We moreover remark that if $\mathfrak{M}$ is a factor then every equivariant γ-inner strictly positive-definite function φ can be renormalized to a state as $\varphi(e) \in \mathcal{Z}(\mathfrak{M}) = \mathbb{C} \cdot \mathbb{1}$. This is however not the case for general $\mathfrak{M}$ as $\varphi(e)$ might not be invertible.

The next theorem shows that the G-equivariant γ-inner $\mathfrak{M}$-valued states are exactly the positive element $f \in \mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)$ satisfying $f(e) = \mathbb{1}$ (up to an involution). This set $\mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)_{e,+}$ has already been discussed in Corollary 4.2.3.

Take $f \in \mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)_{e,+}$ and define $\varphi_f(r) = f(r)^*$. Then $\varphi_f(e) = f(e)^* = \mathbb{1}$ follows directly. A calculation analogous to 4.1.1 shows the G-equivariance

$$\varphi_f(r)^* = f(r) = (\delta_s * \varphi_f * \delta_{s^{-1}})(r) = \gamma_s(f(s^{-1}rs)) = \gamma_s(\varphi_f(s^{-1}rs)^*).$$

The commutation behavior with $\mathfrak{M} \cdot \delta_e$ implies $\varphi_f(r) = f(r)^* \in \mathcal{Z}(\mathfrak{M}, \gamma_r)$ by

$$(\varphi_f(r) \cdot \delta_r)(x \cdot \delta_e) = \varphi_f(r)\gamma_r(x) \cdot \delta_r \overset{!}{=} (x \cdot \delta_e)(\varphi_f(r) \cdot \delta_r) = x\varphi_f(r) \cdot \delta_r.$$

We thus note that the G-equivariance of the function φ_f is equivalent to the commutation with the group G and the γ-innerness $\varphi(s) \in \mathcal{Z}(\mathfrak{M}, \gamma_s)$ is equivalent to the commutation with $\mathfrak{M}$ (up to an involution).

Theorem 4.5.5. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra.*

Then there is an bijection

$$F : \mathcal{S}_\gamma(G, \mathfrak{M}) \to \mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)_{e,+}, \quad F(\varphi)(r) = \varphi(r)^* = \varphi(r^{-1}).$$

Proof. We first prove that $F(\varphi) \in \mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)_{e,+}$. Clearly $F(\varphi)(e) = \varphi(e)^* = \mathbb{1}$. Take $f \in \mathfrak{M} \rtimes_\gamma G$ and compute

$$\begin{aligned}(F(\varphi) * f)(r) &= \sum F(\varphi)(s)\gamma_s(f(s^{-1}r)) = \sum \varphi(s)^*\gamma_s(f(s^{-1}r)) \\ &= \sum \varphi(s^{-1})\gamma_s(f(s^{-1}r)) = \sum f(s^{-1}r)\varphi(s^{-1}) \\ &= \sum f(\tilde{s})\varphi(\tilde{s}r^{-1}) = \sum f(\tilde{s})\gamma_{\tilde{s}}(\varphi(r^{-1}\tilde{s})) \\ &= \sum f(\tilde{s})\gamma_{\tilde{s}}(\varphi(\tilde{s}^{-1}r)^*) = (f * F(\varphi))(r).\end{aligned}$$

This shows that $F(\varphi)$ is central. Note that the correct commutation relation with $\mathfrak{M}$ is due to $\varphi(s) \in \mathcal{Z}(\mathfrak{M}, \gamma_s)$ and the correct commutation relation with G follows from the equivariance relation $\gamma_r(\varphi(s)) = \varphi(rsr^{-1})$.

Explicitly constructing the positive root of $F(\varphi)$ w.r.t. the convolution $*$ via the root of $[\varphi]$ is possible, however convoluted in writing. We therefore take a faithful representation $(\pi, \mathcal{H})$ of $\mathfrak{M}$ and proceed to the induced representation $\hat{\pi} = \pi \rtimes V$ on $L^2(G, \mathcal{H})$ as in Chapter 2. Take a vector-valued function $v \in L^2(G, \mathcal{H})$ and compute

$$\begin{aligned}\langle v, \hat{\pi}(F(\varphi))v\rangle &= \sum \langle v(r), \pi(\varphi(s))^* V_s v(s^{-1}r)\rangle \\ &= \sum \langle v(r), \pi(\varphi(r\tilde{s}^{-1}))^* V_{r\tilde{s}^{-1}} v(\tilde{s})\rangle \\ &= \sum \langle V_r^* v(r), \pi(\varphi(\tilde{s}^{-1}r))^* V_{\tilde{s}}^* v(\tilde{s})\rangle \\ &= \sum \langle V_r^* v(r), \pi(\varphi(r^{-1}\tilde{s})) V_{\tilde{s}}^* v(\tilde{s})\rangle \geq 0.\end{aligned}$$

This shows that $F(\varphi)$ is positive.

We now proceed to show that F is injective and surjective. F is clearly injective by definition. The surjectivity was discussed in a small comment above this theorem. □

This theorem can now be used to connect the extensions $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega$ of a γ-invariant KMS state ω to the crossed product, the set of twisted KMS functionals $\mathcal{F}(\mathcal{A}, \alpha, \gamma)^+_\omega$, the $\mathfrak{M}_\omega$-valued states $\mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega)$ and part of the center of the von Neumann crossed product $\mathcal{Z}(\mathfrak{M} \rtimes_\gamma G)_{e,+}$.

Theorem 4.5.6. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system, $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ and $(\mathfrak{M}_\omega, \overline{\alpha}, \overline{\gamma})$ the corresponding finitely-twisted W^*-dynamical system with standard vector Ω.*

There is the bijection

$$\Xi : \mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega) \to \mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma)^+_\omega, \quad (\Xi\varphi)_s(a) = \langle \Omega, \pi_\omega(a)\varphi(s)\Omega\rangle.$$

The following sets are in convex bijective correspondence:

1) $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega$;

2) $\mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma)^+_\omega$;

3) $\mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega)$;

4) $\mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)_{e,+}$.

Before giving the proof of this theorem, we write down a corollary which describes the case that $\hat{\omega}^{\mathrm{can}}$ is the unique extension of a γ-invariant KMS state ω on $(\mathcal{A}, \alpha, \gamma)$.

Corollary 4.5.7. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system, $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ and $(\mathfrak{M}_\omega, \overline{\alpha}, \overline{\gamma})$ the corresponding finitely-twisted W^*-dynamical system with standard vector Ω.*

The following are equivalent:

1) $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega = \{\hat{\omega}^{\mathrm{can}}\}$;

2) $\mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma)^+_\omega$ *consists only of the trivial family of* ω;

3) $\mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega) = \{\varphi_{\mathrm{triv}}\}$;

4) $\mathcal{Z}(\mathfrak{M}_\omega \rtimes_{\overline{\gamma}} G)_{e,+} = \{\varphi_{\mathrm{triv}}\}$.

Proof. We begin the proof by showing that every state $\varphi \in \mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega)$ defines an equivariant and positively-compatible family of twisted KMS functionals. Clearly, $\Xi(\varphi)_e(a) = \langle \Omega, \pi_\omega(a)\Omega\rangle = \omega(a)$. Since φ is equivariant and ω is γ-invariant, it follows

$$\begin{aligned}(\Xi\varphi)_{srs^{-1}}(a) &= \langle \Omega, \pi_\omega(a)\varphi(srs^{-1})\Omega\rangle = \langle \Omega, \pi_\omega(a)\gamma_s(\varphi(r))\Omega\rangle \\ &= \langle \Omega, \pi_\omega(\gamma_{s^{-1}}(a))\varphi(r)\Omega\rangle = (\Xi\varphi)_r(\gamma_{s^{-1}}(a)).\end{aligned}$$

This shows the equivariance of the family $(\Xi\varphi)_r$. The functionals $\Xi(\varphi)_s$ are twisted KMS as they are given by an operator $\varphi(s) \in \mathfrak{M}_\omega$ which satisfies equation (4.3.6) by Proposition 4.4.5.

Combining these properties with the positivity of φ shows the positive-compatibility of the family $(\Xi\varphi)_r$: Take a family $(a_s)_{s\in G}$ of elements in $\mathcal{A}$ and compute

$$\begin{aligned}(\Xi\varphi)_{rs^{-1}}(a_s^* a_r) &= \langle \Omega, \pi_\omega(a_s^* a_r)\varphi(rs^{-1})\Omega\rangle = \langle \Omega, \pi_\omega(a_s^*)\varphi(rs^{-1})\overline{\gamma}_{sr^{-1}}(\pi_\omega(a_r))\Omega\rangle \\ &= \langle V_s^*\pi_\omega(a_s)\Omega, \varphi(sr^{-1})V_r^*\pi_\omega(a_r)\Omega\rangle.\end{aligned}$$

Taking the sum over s and r and using the positivity of φ shows that the matrix $[(\Xi\varphi)_{rs^{-1}}(a_s^* a_r)]_{r,s\in G}$ is positive-semidefinite.

On the converse, every family $(\omega_s)_{s\in G} \in \mathcal{F}_\beta(\mathcal{A}, \alpha, \gamma)^+$ with $\omega_e = \omega$ defines a family of operators $(x_s)_{s\in G}$ by Proposition 4.3.5, where $x_s \in \mathcal{Z}(\mathfrak{M}_\omega, \overline{\gamma}_s)$ by Proposition 4.4.5.

The above calculations for equivariance and positivity can then be repeated with the function $\varphi(s) := x_s$ and show that $\varphi \in \mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega)$. Moreover, these constructions are clearly their respective inverse. The map Ξ is moreover convex as for $\lambda \in (0,1)$ and $\varphi_1, \varphi_2 \in \mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega)$ it holds

$$\Xi(\lambda\varphi_1 + (1-\lambda)\varphi_2)_s = \langle \Omega, \pi_\omega(\cdot)(\lambda\varphi_1 + (1-\lambda)\varphi_2)(s)\Omega \rangle = \lambda\Xi(\varphi_1)_s + (1-\lambda)\Xi(\varphi_2)_s.$$

We now prove the second claim of the theorem.

1) $\simeq$ 2): This is a consequence of Theorem 4.3.1. Restricting to these $\hat{\omega} \in \mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ which pull back to ω gives families $(\omega_s)_{s\in G}$ where $\omega_e = \omega$ and vice versa.

2) $\simeq$ 3): This is the result of the first part of this theorem.

3) $\simeq$ 4): This is shown in Theorem 4.5.5. □

This theorem shows that there are different ways of thinking about an extension $\hat{\omega}$ of a KMS state $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ to the crossed product $\mathcal{A} \rtimes_\gamma G$, namely via families of twisted KMS functionals and via $\mathfrak{M}_\omega$-valued states. These approaches do however have similarities. There is always a KMS component to them, which stems from the KMS property of the state $\hat{\omega}$ on $\mathcal{A} \rtimes_\gamma G$. The family of functionals satisfies γ_s-twisted KMS conditions and the state $\varphi \in \mathcal{S}(G, \mathfrak{M}_\omega)$ takes values in the twisted centers. They both incorporate a type of G-equivariance, namely $\omega_s \circ \gamma_r = \omega_{r^{-1}sr}$ and $\overline{\gamma}_r(\varphi(s)) = \varphi(rsr^{-1})$. This is related to the $\hat{\alpha}$-invariance of $G \subset \mathcal{A} \rtimes_\gamma G$ and the KMS property of $\hat{\omega}$. The positivity of $\hat{\omega}$ is reflected in the positive compatibility of the family of functionals $(\omega_s)_{s\in G}$ as well as in the positivity of φ. These three properties are all of a different nature and especially the positivity is independent from the KMS type properties.

In practice it is best to combine the different methods, as some method might be better adapted to the finitely-twisted C^*-dynamical system under consideration. Regarding the example of the CAR algebra studied in Chapter 5, it is useful to compute the form of a twisted KMS functional ρ and then require it to be dominated by the unique KMS state ω of the system. This domination requirement already shows that ρ has to vanish if certain assumptions on α and γ are not valid. The existence of the functional under these assumptions is best proven either via Proposition 4.4.7 or by explicitly showing that the non-commutative Radon-Nikodým derivative exists.

We study the structure of the set of extensions $\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega$ in the next section. This is done in the language of G-equivariant γ-inner states on G.

4.6 The Structure of Equivariant γ-Inner $\mathfrak{M}$-valued States

Theorem 4.5.6 asserts that the extensions of a KMS state ω on $\mathcal{A}$ to the crossed product correspond to the equivariant $\overline{\gamma}$-inner $\mathfrak{M}_\omega$-valued states. In this section, we therefore analyze the structure of $\mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_\omega)$. In the case that the action is (weakly) inner, a nice decomposition in terms of the characters of G is possible.

Two $\mathfrak{M}$-valued states $\varphi, \psi \in \mathcal{S}(G, \mathfrak{M})$ can be "multiplied" in a number of different ways. Some of these multiplications do however not leave $\mathcal{S}(G, \mathfrak{M})$ invariant, which we examine in the following.

Definition 4.6.1. *Let $\mathfrak{M}$ be a von Neumann algebra, G a finite group and $\varphi, \psi \in \mathcal{S}(G, \mathfrak{M})$. We then define the following multiplications:*

1) *The pointwise product $(\varphi \cdot \psi)(r) := \varphi(r)^*\psi(r)$;*

2) *The convolution product $(\varphi * \psi)(r) := \sum \varphi(s)\psi(s^{-1}r)$;*

3) *The γ-twisted convolution product $(\varphi *_\gamma \psi)(r) := \sum \varphi(s)\gamma_s(\psi(s^{-1}r))$;*

4) *The $\mathfrak{M}$-valued scalar product $\langle \varphi, \psi \rangle := \sum \varphi(s)^*\psi(s)$;*

where $(\mathfrak{M}, \gamma)$ is assumed to be a finitely-twisted von Neumann algebra for item 3).

The pointwise product $\cdot$ is chosen in such a way that for $\varphi, \psi \in \mathcal{S}_\gamma(G, \mathfrak{M})$, the product $\varphi \cdot \psi$ is $\mathcal{Z}(\mathfrak{M})^\gamma$-valued. Note that the convolution product $\varphi * \psi$ of two $\mathfrak{M}$-valued states fits the multiplication of the matrices $[\varphi]$ and $[\psi]$ in the following sense

$$\begin{aligned}[\varphi * \psi]_{r,s} = (\varphi * \psi)(rs^{-1}) &= \sum_k \varphi(k)\psi(k^{-1}rs^{-1}) = \sum_k \varphi(rk^{-1})\psi(ks^{-1}) \\ &= \sum_k [\varphi]_{r,k}[\psi]_{k,s} = ([\varphi][\psi])_{r,s}.\end{aligned}$$

Similarly, item 3) is exactly the convolution multiplication of φ and ψ in $\mathfrak{M} \rtimes_\gamma G$.

The pointwise multiplication of two $\mathfrak{M}$-valued states is in general not an $\mathfrak{M}$-valued state as the positivity might fail. The positivity however still holds in case the $\mathfrak{M}$-valued states commute pointwise.

Lemma 4.6.2. *Let $\mathfrak{M}$ be a von Neumann algebra, G a finite group and $\varphi, \psi \in \mathcal{S}(G, \mathfrak{M})$.*

If φ, ψ pointwise commute, then $\varphi \cdot \psi \in \mathcal{S}(G, \mathfrak{M})$.

Proof. As φ is a state, so is φ^* and we replace φ^* by φ for the following argument. If φ and ψ commute pointwise, then they lie in commuting von Neumann subalgebras $\mathfrak{M}_\varphi$ and $\mathfrak{M}_\psi$ of $\mathfrak{M}$. Thus $[\varphi] \in M_{|G|}(\mathfrak{M}_\varphi)$ has a root $\sqrt{[\varphi]}$ that commutes with $[\psi] \in M_{|G|}(\mathfrak{M}_\psi)$ pointwise. Passing to a representation, this shows for $w \in \mathcal{H}^{|G|}$

$$\begin{aligned}
\langle w, [\varphi \cdot \psi] w\rangle &= \sum_{s,r} \langle w_r, [\varphi]_{rs}[\psi]_{rs} w_s\rangle \\
&= \sum_{s,r,k} \langle w_r, \sqrt{[\varphi]}_{rk}\sqrt{[\varphi]}_{ks}[\psi]_{rs} w_s\rangle \\
&= \sum_{s,r,k} \langle (\sqrt{[\varphi]}_{rk})^* w_r, [\psi]_{rs}\sqrt{[\varphi]}_{ks} w_s\rangle \\
&= \sum_{k}\sum_{s,r} \langle \sqrt{[\varphi]}_{kr} w_r, [\psi]_{rs}\sqrt{[\varphi]}_{ks} w_s\rangle \\
&= \sum_{k}\sum_{s,r} \langle v_{r,k}, [\psi]_{rs} v_{s,k}\rangle \geq 0,
\end{aligned}$$

as a sum of positive terms, where $v_{s,k} = \sqrt{[\varphi]}_{ks} w_s$. Therefore, $\varphi \cdot \psi$ is positive and clearly $(\varphi \cdot \psi)(e) = \mathbb{1}$. □

The (γ-twisted) convolution of two $\mathfrak{M}$-valued states is in general not normalized and therefore not an $\mathfrak{M}$-valued state but only an $\mathfrak{M}$-valued positive-definite function. As an example consider $G = \mathbb{Z}_2 = \{0, 1\}$, $\mathfrak{M} = \mathbb{C}$ and the two characters χ_0 and χ_1 of $\mathbb{Z}_2$. It then holds

$$\begin{aligned}
\chi_0 * \chi_1(0) &= \chi_0(0)\chi_1(0) + \chi_0(1)\chi_1(1) = 1 + (-1) = 0, \\
\chi_0 * \chi_1(1) &= \chi_0(0)\chi_1(1) + \chi_0(1)\chi_1(0) = (-1) + 1 = 0.
\end{aligned}$$

Thus $\varphi * \psi(e)$ is in general not invertible and can therefore not be normalized to a state.

From now on we focus on the multiplication $\cdot$, as the product of two commuting $\mathfrak{M}$-valued states is again an $\mathfrak{M}$-valued state. We are moreover going to assume that γ is inner in $\mathfrak{M}$.

Proposition 4.6.3. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra and γ inner in $\mathfrak{M}$ via the unitary representation $\mathcal{V}$.*

Then

$$\mathcal{S}_\gamma(G, \mathfrak{M}) = \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M})) \cdot \mathcal{V} \quad \textit{and} \quad \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M})) = \mathcal{S}_\gamma(G, \mathfrak{M}) \cdot \mathcal{V}.$$

Note that this proposition is similar to the Kallman decomposition 4.4.8 for the case that all Kallman projections coincide with $\mathbb{1}$.

Proof. First note that $\mathcal{V} \in \mathcal{S}_\gamma(G, \mathfrak{M})$ by Lemma 4.5.4. That is, every $\mathfrak{M}$-valued unitary representation is an $\mathfrak{M}$-valued state, equivariant and $\mathcal{V}(s) \in \mathcal{Z}(\mathfrak{M}, \gamma_s)$.

Take $\varphi \in \mathcal{S}_\gamma(G, \mathfrak{M})$ and consider the product $\varphi \cdot \mathcal{V}$, which is clearly $\mathcal{Z}(\mathfrak{M})$-valued and $\varphi \cdot \mathcal{V}(e) = \mathbb{1}$. We prove positivity of $\varphi \cdot \mathcal{V}$ by passing to a representation and choosing a family $(v_s)_{s\in G}$ of vectors and defining $v \in \mathcal{H}^{|G|}$ via this family. It follows

$$\begin{aligned}\langle v, [\varphi \cdot \mathcal{V}]v\rangle &= \sum \langle v_s, [\varphi \cdot \mathcal{V}]_{s,r} v_r\rangle = \sum \langle v_s, \varphi(sr^{-1})^* \mathcal{V}(sr^{-1}) v_r\rangle \\ &= \sum \langle \mathcal{V}(s)^* v_s, \gamma_{s^{-1}}(\varphi(rs^{-1}))\mathcal{V}(r)^* v_r\rangle \\ &= \sum \langle \mathcal{V}(s)^* v_s, \varphi(s^{-1}r)\mathcal{V}(r)^* v_r\rangle \\ &= \sum \langle w_s, \varphi(s^{-1}r) w_r\rangle \geq 0.\end{aligned}$$

This calculation heavily relies on $\mathcal{V}$ being a representation of G and $\varphi \in \mathcal{S}_\gamma(G, \mathfrak{M})$. It remains to show that $\varphi \cdot \mathcal{V}$ is constant on conjugacy classes. As $\varphi(r)^*\mathcal{V}(r) \in \mathcal{Z}(\mathfrak{M})$ for all $r \in G$ it follows

$$\begin{aligned}(\varphi \cdot \mathcal{V})(srs^{-1}) &= \varphi(srs^{-1})^* \mathcal{V}(srs^{-1}) = \gamma_s(\varphi(r)^*\mathcal{V}(r)) = \mathcal{V}(s)(\varphi(r)^*\mathcal{V}(r))\mathcal{V}(s)^* \\ &= \varphi(r)^*\mathcal{V}(r) = (\varphi \cdot \mathcal{V})(r).\end{aligned}$$

This shows $\mathcal{S}_\gamma(G, \mathfrak{M}) \cdot \mathcal{V} \subset \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M}))$.

Take now $z \in \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M}))$ and consider $z \cdot \mathcal{V}$. Clearly, $z \cdot \mathcal{V}$ is normalized, equivariant and $z \cdot \mathcal{V}(r) \in \mathcal{Z}(\mathfrak{M}, \gamma_r)$. The positivity is shown by a calculation analogous to the one above. This shows $\mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M})) \cdot \mathcal{V} \subset \mathcal{S}_\gamma(G, \mathfrak{M})$. Clearly $(\varphi \cdot \mathcal{V}) \cdot \mathcal{V} = \varphi$, which concludes the proof. □

In the case that $\mathfrak{M}$ is a factor, $\mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M})) = \mathcal{S}_c(G)$ are just the states on G that are constant on conjugacy classes. This allows for the connection to the theory of characters of finite groups. Recall that a character of a finite group G has as data a finite dimensional representation $(\pi, \mathcal{K})$ of G. The associated (normalized) character $\chi_{(\pi,\mathcal{K})}$ is then defined via

$$\chi_{(\pi,\mathcal{K})}(s) := \dim_{\mathbb{C}}(\mathcal{K})^{-1} \operatorname{Tr}_{\mathcal{K}} \pi(s)$$

and the set of all characters is denoted $\operatorname{Char}(G)$. As every representation of a finite group is unitarizable, we can furthermore assume that $(\pi, \mathcal{K})$ is a unitary representation. One of the main results of representation theory of finite groups is then that the set of characters of irreducible representations (up to equivalence) forms a basis of the vector space of class functions. Here a class function is a function $f : G \to \mathbb{C}$ which is constant on conjugacy classes of G. There is only a finite number of irreducible representations of a finite group as

$$|G| = \dim_{\mathbb{C}}(\mathcal{K}_1)^2 + \ldots + \dim_{\mathbb{C}}(\mathcal{K}_n)^2,$$

where $(\pi_i, \mathcal{K}_i)_{i=1,\ldots,n}$ is a system of representatives of irreps of G.

Moreover, every character $\chi_{(\pi,\mathcal{K})}$ is a state on G. For a family $(\lambda_s)_{s\in G}$ in $\mathbb{C}$

$$\sum \lambda_s \overline{\lambda_r} \chi_{(\pi,\mathcal{K})}(sr^{-1}) = \sum \lambda_s \overline{\lambda_r} \operatorname{Tr}_{\mathcal{K}}(\pi(s)\pi(r)^*) = \operatorname{Tr}_{\mathcal{K}}\Big((\sum_s \lambda_s \pi(s))(\sum_r \lambda_r \pi(r))^*\Big) \geq 0$$

shows the positivity of $\chi_{(\pi,\mathcal{K})}$ up to normalization. Regarding the convex structure of $\mathcal{S}_c(G)$, characters of irreducible representations are exactly the extreme points of $\mathcal{S}_c(G)$. Take $(\pi, \mathcal{K})$ irreducible and decompose $\chi_{(\pi,\mathcal{K})} = \lambda\omega_1 + (1-\lambda)\omega_2$ in $\mathcal{S}_c(G)$. Then $\lambda^{-1}\chi_{(\pi,\mathcal{K})} - \omega_1 \geq 0$ as a positive functional. This allows one to introduce, similarly to Proposition 4.3.5, an operator $T_1 \in \pi(G)'$ implementing ω_1. As π is irreducible it follows $T_1 = \mathbb{1}$ and $\omega_1 = \omega_2 = \chi_{(\pi,\mathcal{K})}$, which is the extremality of $\chi_{(\pi,\mathcal{K})}$. On the converse, a reducible character can always be convexly decomposed into irreducible ones. Therefore, $\mathcal{S}_c(G)$ is the convex hull of the set of irreducible characters $\operatorname{IrrChar}(G)$.

Let us collect these findings in a lemma.

Lemma 4.6.4. *Let G be a finite group.*

Then

$$\mathcal{S}_c(G) = \operatorname{Char}(G) \quad \textit{and} \quad \partial_e \operatorname{Char}(G) = \operatorname{IrrChar}(G)$$

and $\partial_e \operatorname{Char}(G)$ is finite.

Combining these results on characters of finite groups with Proposition 4.6.3 yields the following corollary.

Corollary 4.6.5. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra, $\mathfrak{M}$ a factor and γ inner in $\mathfrak{M}$ via the unitary representation $\mathcal{V}$.*

Then

$$\mathcal{S}_\gamma(G, \mathfrak{M}) = \operatorname{Char}(G) \cdot \mathcal{V} \quad \textit{and} \quad \partial_e \mathcal{S}_\gamma(G, \mathfrak{M}) = \partial_e \operatorname{Char}(G) \cdot \mathcal{V}.$$

This corollary fully characterizes the equivariant γ-inner states on a factor $\mathfrak{M}$ in case the group action is inner. As a direct consequence, the extensions of extremal KMS states to $\mathcal{A} \rtimes_\gamma G$ are characterized. This shows in particular that there are as many extremal extensions of ω to $\mathcal{A} \rtimes_\gamma G$ as there are irreducible representations of G.

Corollary 4.6.6. *Let $(\mathcal{A}, \alpha, \gamma)$ be a finitely-twisted C^*-dynamical system and $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$ extremal.*

If γ is ω-weakly inner with unitary representation $\overline{\mathcal{V}} : G \to \mathfrak{M}_\omega$, then

$$\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega \simeq \operatorname{Char}(G) \quad \textit{and} \quad \partial_e(\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega) \simeq \operatorname{IrrChar}(G).$$

In case ω is not extremal, the correspondence with the characters can not be drawn. But still, the following relation holds

$$\mathcal{S}_\beta(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})_\omega \simeq \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M}_\omega)) \cdot \overline{\mathcal{V}}.$$

Proof. This is a direct consequence of Theorem 4.5.6, Proposition 4.6.3 and Lemma 4.6.4. □

In case the group G is abelian, we make the following observations

$$\varphi(r)\psi(s) = \gamma_r(\psi(s))\varphi(r) = \psi(rsr^{-1})\varphi(r) = \psi(s)\varphi(r),$$

for $\varphi, \psi \in \mathcal{S}_\gamma(G, \mathfrak{M})$. In particular Lemma 4.6.2 can be applied to equivariant γ-inner $\mathfrak{M}$-valued states over abelian groups.

Corollary 4.6.7. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra and assume that G is abelian. If $\varphi, \psi \in \mathcal{S}_\gamma(G, \mathfrak{M})$, then φ and ψ commute pointwise.*

In particular $\varphi \cdot \psi \in \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M}))$ and the map

$$\cdot : \mathcal{S}_\gamma(G, \mathfrak{M}) \times \mathcal{S}_\gamma(G, \mathfrak{M}) \to \mathcal{S}_c(G, \mathcal{Z}(\mathfrak{M})), \quad (\varphi, \psi) \mapsto \varphi \cdot \psi$$

is well-defined.

Corollary 4.6.8. *Let $(\mathfrak{M}, \gamma)$ be a finitely-twisted von Neumann algebra, where $\mathfrak{M}$ is a factor and G is simple and abelian.*

Then one of the following is true:

1) *γ is inner and $\mathcal{S}_\gamma(G, \mathfrak{M}) = \mathrm{Conv}(\hat{G}) \cdot \mathcal{V}$ or;*
2) *γ is outer and $\mathcal{S}_\gamma(G, \mathfrak{M}) = \{\varphi_{\mathrm{triv}}\}$.*

Here $\hat{G}$ denotes the dual group of G.

Proof. By Corollary 4.4.10, we have that γ is either inner or properly outer. If γ is inner with unitary representation $\mathcal{V}$, we have

$$\mathcal{S}_\gamma(G, \mathfrak{M}) = \mathrm{Conv}(\mathrm{IrrChar}(G)) \cdot \mathcal{V} = \mathrm{Conv}(\hat{G}) \cdot \mathcal{V}.$$

If γ is properly outer, then every γ-inner state φ satisfies $\varphi(e) = \mathbb{1}, \varphi(s) = 0$ for $s \neq e$. These conditions already determine φ uniquely. □

4.7 Asymptotically Abelian Dynamics

We now shift the focus from the twist γ to the dynamics α and discuss weak asymptotic commutativity, as it plays a prominent role in many physical models [BR97; BL99]. There is also a graded version of asymptotic commutativity which will however not be covered in this thesis, see [BL99; SGL24]. Note that asymptotic abelianess is a distinct feature of the theory of C^*-dynamical systems.

Definition 4.7.1. *A C^*-dynamical system $(\mathcal{A}, \alpha)$ is called ω-weakly asymptotically abelian for $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$, if for all $a, b, c \in \mathcal{A}$*

$$\omega(a[b, \alpha_t(c)]) \xrightarrow[t\to\infty]{} 0.$$

A W^*-dynamical system $(\mathfrak{M}, \alpha)$ is called asymptotically abelian, if for all $x, y \in \mathfrak{M}$

$$[x, \alpha_t(y)] \xrightarrow[t\to\infty]{WOT} 0.$$

When interpreting a C^*-algebra $\mathcal{A}$ as the C^*-dynamical system with trivial dynamics, then the KMS states are exactly the traces τ. A τ-weakly asymptotically abelian C^*-algebra is abelian, assuming τ is faithful.

The following lemma states that weak asymptotically abelianess gets passed on to the enveloping von Neumann algebra in the GNS representation. The proof is adapted from the version of graded asymptotic abelianess discussed in [BL99, Sec. 3].

Lemma 4.7.2. *Let $(\mathcal{A}, \alpha)$ be a ω-weakly asymptotically abelian C^*-dynamical system for $\omega \in \mathcal{S}_\beta(\mathcal{A}, \alpha)$.*

Then the corresponding W^*-dynamical system $(\mathfrak{M}_\omega, \overline{\alpha})$ is asymptotically abelian.

Proof. Consider the GNS triple $(\pi, \mathcal{H}, \Omega)$ of ω. Let $a, b, c \in \mathcal{A}$ then

$$\langle \pi(a)\Omega, [\pi(b), \overline{\alpha}_t(\pi(c))]\Omega\rangle = \omega(a^*[b, \alpha_t(c)]) \xrightarrow[t\to\infty]{} 0.$$

Since $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$, $[\pi(b), \overline{\alpha}_t(\pi(c))]\Omega$ converges weakly to 0. It then follows for $x', y' \in \mathfrak{M}'$

$$\langle x'\Omega, [\pi(b), \overline{\alpha}_t(\pi(c))]y'\Omega\rangle = \langle y'^* x'\Omega, [\pi(b), \overline{\alpha}_t(\pi(c))]\Omega\rangle \xrightarrow[t\to\infty]{} 0$$

and since Ω is cyclic for $\mathfrak{M}'$, $[\pi(b), \overline{\alpha}_t(\pi(c))]$ converges weakly to 0 as well.

This can now be lifted to a similar statement on $\mathfrak{M}$ by Kaplansky's density theorem. The unit ball $\pi(\mathcal{A})_1$ of $\pi(\mathcal{A})$ is dense in the unit ball $\mathfrak{M}_1$ of $\mathfrak{M}$ in the strong-* operator topology. Thus for $x, y \in \mathfrak{M}$ and $\epsilon > 0$ there exist $\pi(a), \pi(b) \in \pi(\mathcal{A})_1$ s.t.

$$\|(x - \pi(a))\Omega\|, \|(x^* - \pi(a^*))\Omega\|, \|(y - \pi(b))\Omega\|, \|(y^* - \pi(b)^*)\Omega\| < \epsilon.$$

By the following calculation also $[x, \overline{\alpha}_t(y)]$ converges to 0 weakly. Take $z' \in \mathfrak{M}'_1$

$$\begin{aligned}
&|\langle [x, \overline{\alpha}_t(y)]\Omega, z'\Omega\rangle - \langle [\pi(a), \overline{\alpha}_t(\pi(b))]\Omega, z'\Omega\rangle| \\
&\leq |\langle (x\overline{\alpha}_t(y) - \pi(a)\overline{\alpha}_t(\pi(b)))\Omega, z'\Omega\rangle| + |\langle (\overline{\alpha}_t(y)x - \overline{\alpha}_t(\pi(b))\pi(a))\Omega, z'\Omega\rangle|
\end{aligned}$$

Upper bounds for both terms on the right can be found by calculations of the sort

$$\begin{aligned}
&|\langle (\overline{\alpha}_t(y)x - \overline{\alpha}_t(\pi(b))\pi(a))\Omega, z'\Omega\rangle| \\
&= |\langle (\overline{\alpha}_t(y - \pi(b))x - \overline{\alpha}_t(\pi(b))(\pi(a) - x))\Omega, z'\Omega\rangle| \\
&\leq |\langle \overline{\alpha}_t(y - \pi(b))x\Omega, z'\Omega\rangle| + |\langle \overline{\alpha}_t(\pi(b))(\pi(a) - x)\Omega, z'\Omega\rangle| \\
&= |\langle z'^* x\Omega, \overline{\alpha}_t(y^* - \pi(b^*))\Omega\rangle| + |\langle (\pi(b) - y))\Omega, \overline{\alpha}_t(\pi(b^*))z'\Omega\rangle| < 2\epsilon,
\end{aligned}$$

where the Cauchy-Schwarz inequality and the fact that all operators are contained in either $\mathfrak{M}_1$ or $\mathfrak{M}_1'$ was used. This similarly works for the other term. As ϵ was arbitrary and $[\pi(a), \overline{\alpha}_t(\pi(b))]\Omega$ converges weakly to 0, so does

$$[x, \overline{\alpha}_t(y)]\Omega \xrightarrow[t\to\infty]{} 0.$$

By the same argument as above, also the operator $[x, \overline{\alpha}_t(y)]$ vanishes weakly. Therefore, the von Neumann algebra $\mathfrak{M}_\omega = \mathfrak{M}$ is asymptotically abelian. □

We derive strong implications of asymptotically abelian dynamics for the twisted center in Lemma 4.7.4 and Corollary 4.7.5. As a starting point, if the twisted center $\mathcal{Z}(\mathfrak{M}, \gamma)$ is contained in the center $\mathcal{Z}(\mathfrak{M})$ and γ is non-trivial, then $\mathcal{Z}(\mathfrak{M}, \gamma)$ vanishes.

Lemma 4.7.3. *Let $\mathfrak{M}$ be a von Neumann algebra, $\gamma \in \mathrm{Aut}(\mathfrak{M})$ and suppose $\gamma|_{\mathfrak{M}_p} \neq \mathbb{1}_{\mathfrak{M}_p}$ for all non-zero γ-invariant central projections $p \in \mathcal{Z}(\mathfrak{M})^\gamma$.*

If $\mathcal{Z}(\mathfrak{M}, \gamma) \subset \mathcal{Z}(\mathfrak{M})$, then $\mathcal{Z}(\mathfrak{M}, \gamma) = 0$.

Note that this lemma in particular applies if $\mathfrak{M}$ is a factor and $\gamma \neq \mathbb{1}$.

Proof. By Corollary 4.4.4, $\mathcal{Z}(\mathfrak{M}, \gamma) = v \cdot \mathcal{Z}(\mathfrak{M}_p)^\gamma \oplus 0$, where p is a γ-invariant central projection and $\gamma|_{\mathfrak{M}_p} = \mathrm{Ad}_v|_{\mathfrak{M}_p}$. By the assumption that $\mathcal{Z}(\mathfrak{M}, \gamma)$ is contained in the center, it follows $v \in \mathcal{Z}(\mathfrak{M})$. This implies directly $\gamma|_{\mathfrak{M}_p} = \mathbb{1}|_{\mathfrak{M}_p}$, which contradicts the non-triviality of γ. Thus $p = 0$. □

If the asymptotically abelian von Neumann algebra furthermore has a standard vector, the following lemma can be derived.

Lemma 4.7.4. *Let $(\mathfrak{M}, \alpha)$ be an asymptotically abelian W^*-dynamical system, Ω a standard vector s.t. α is the modular dynamics of Ω. Let $\gamma \in \mathrm{Aut}(\mathfrak{M})$ and assume ω is γ-invariant.*

The following statements holds:

1) $\mathfrak{M}^\alpha = \mathcal{Z}(\mathfrak{M})$ *and in particular* $\mathcal{Z}(\mathfrak{M}, \gamma) \subset \mathcal{Z}(\mathfrak{M})^\gamma$;

2) *If* $\gamma|_{\mathfrak{M}_p} \neq \mathbb{1}_{\mathfrak{M}_p}$ *for all non-zero γ-invariant central projections* $p \in \mathcal{Z}(\mathfrak{M})^\gamma$, *then* $\mathcal{Z}(\mathfrak{M}, \gamma) = \{0\}$;

3) *If* $\mathcal{Z}(\mathfrak{M})^\gamma = \mathbb{C} \cdot \mathbb{1}$ *and* $\gamma \neq \mathbb{1}$, *then* $\mathcal{Z}(\mathfrak{M}, \gamma) = 0$.

Proof. 1): The inclusion $\mathcal{Z}(\mathfrak{M}) \subset \mathcal{Z}_\Omega(\mathfrak{M}) = \mathfrak{M}^\alpha$ is evident. Thus, we show the opposite inclusion. Consider $x \in \mathcal{Z}_\Omega(\mathfrak{M}) = \mathfrak{M}^\alpha$ and $y \in \mathfrak{M}$, then

$$[y, x] = [y, \alpha_t(x)] \xrightarrow[t\to\infty]{} 0.$$

The left hand side is independent of t and thus $x \in \mathcal{Z}(\mathfrak{M})$. The inclusion $\mathcal{Z}(\mathfrak{M}, \gamma) \subset \mathcal{Z}(\mathfrak{M})^{\gamma}$ now follows from Lemma 4.4.6.

2) and 3): These follow directly from Corollary 4.7.3 in combination with 1). □

This argument can now be lifted to a finitely-twisted von Neumann algebra with standard vector.

Corollary 4.7.5. *Let $(\mathfrak{M}, \alpha, \gamma)$ be a asymptotically abelian finitely-twisted W^*-dynamical system with standard vector Ω s.t. α is the modular dynamics of Ω and the associated state ω is γ-invariant.*

The following statements holds:

1) $\mathfrak{M}^{\alpha} = \mathcal{Z}(\mathfrak{M})$ *and in particular* $\mathcal{Z}(\mathfrak{M}, \gamma_s) \subset \mathcal{Z}(\mathfrak{M})^{\gamma_s}$ *for all* $s \in G$*;*
2) *If for all* $s \in G\backslash\{e\}$*,* $\gamma_s|_{\mathfrak{M}_p} \neq \mathbb{1}_{\mathfrak{M}_p}$ *for all non-zero* γ_s*-invariant central projections* $p \in \mathcal{Z}(\mathfrak{M})$*, then* $\mathcal{S}_{\gamma}(G, \mathfrak{M}) = \{\varphi_{\mathrm{triv}}\}$*;*
3) *If* $\mathfrak{M}$ *is a factor and* $\gamma_s \neq \mathbb{1}$ *for all* $s \in G\backslash\{e\}$*, then* $\mathcal{S}_{\gamma}(G, \mathfrak{M}) = \{\varphi_{\mathrm{triv}}\}$*;*
4) *If* $\mathcal{Z}(\mathfrak{M})^{\gamma_s} = \mathbb{C} \cdot \mathbb{1}$ *and* $\gamma_s \neq \mathbb{1}$ *for all* $s \in G\backslash\{e\}$*, then* $\mathcal{S}_{\gamma}(G, \mathfrak{M}) = \{\varphi_{\mathrm{triv}}\}$*;*

Proof. 1) and 2): These follow directly by applying Lemma 4.7.4 to each γ_s for $s \neq e$. If $\mathcal{Z}(\mathfrak{M}, \gamma_s) = \{0\}$ for $s \neq e$, then every γ-inner state φ satisfies $\varphi(e) = \mathbb{1}, \varphi(s) = 0$ for $s \neq e$. This shows $\varphi = \varphi_{\mathrm{triv}}$.

3): This follows directly from 2).

4): Item 3) of the above corollary can be applied and shows $\mathcal{Z}(\mathfrak{M}, \gamma_s) = 0$ for $s \neq e$. □

It is not sufficient for item 4) to assume that $\mathcal{Z}(\mathfrak{M})^{\gamma}$ is trivial. Namely, it is not the case that $x \in \mathcal{Z}(\mathfrak{M}, \gamma_s)$ is invariant under all automorphisms γ_r.

As a counterexample consider an asymptotically abelian factor $(\mathcal{N}, \alpha)$ with standard vector Ω and define

$$\mathfrak{M} = \mathcal{N} \oplus \mathcal{N} \oplus \mathcal{N}, \quad \hat{\alpha}_t(x \oplus y \oplus z) = \alpha_t(x) \oplus \alpha_t(y) \oplus \alpha_t(z), \quad \hat{\Omega} = \frac{1}{\sqrt{3}} \Omega \oplus \Omega \oplus \Omega$$

with the symmetric group of order 3 acting by permutations. As $\mathcal{N}$ is asymptotically abelian, it follows that $\mathfrak{M}$ is asymptotically abelian with $\hat{\Omega}$ standard. The S_3-invariant part of the center of $\mathfrak{M}$ is clearly $\mathbb{C} \cdot \mathbb{1}$, whereas $0 \oplus 0 \oplus \mathbb{1}$ is the Kallman projection of the transposition τ_{12}. This projection is not invariant under S_3 but the transposition τ_{12} is non-trivial.

These results can again be applied to finitely-twisted C^*-dynamical systems with ω-weakly asymptotically abelian dynamics. In particular, the following corollary holds by a combination of Theorem 4.5.6 and Corollary 4.7.5.

Corollary 4.7.6. *Let $(\mathcal{A}, \alpha, \gamma)$ be a ω-weakly asymptotically abelian finitely-twisted C^*-dynamical system for $\omega \in \partial_e \mathcal{S}_\beta(\mathcal{A}, \alpha)$ and γ-invariant.*

If $\gamma_s \neq \mathbb{1}$ for $s \in G \backslash \{e\}$, then $\hat{\omega}^{\mathrm{can}}$ is the unique extension of ω to $\mathcal{A} \rtimes_\gamma G$.

This applies in particular if ω is the unique (α, β)-KMS state of the ω-weakly asymptotically abelian system $(\mathcal{A}, \alpha, \gamma)$.

Concluding this chapter, we have shown that every γ-invariant KMS state ω of a finitely-twisted C^*-dynamical system can be extended to the crossed product $(\mathcal{A} \rtimes_\gamma G, \hat{\alpha})$ by considering $\hat{\omega}^{\mathrm{can}}$. On the converse, KMS states that are not γ-invariant can not be extended. The extension $\hat{\omega}^{\mathrm{can}}$ is in general not unique. The non-uniqueness depends on the interplay between a number of factors: The extremality of ω in $\mathcal{S}_\beta(\mathcal{A}, \alpha)^\gamma$, the properties of α and the innerness of $\overline{\gamma}$ in $\mathfrak{M}_\omega$. The main tools for studying these relations are the twisted functionals on $\mathcal{A}$ and the twisted center of $\mathfrak{M}_\omega$.

As for the physical interpretation, an observable algebra $\mathcal{A}$ with time evolution α and thermal equilibrium state ω can be enlarged by a (finite) symmetry group G. The resulting enlarged observable algebra $\mathcal{A} \rtimes G$ allows for an extended thermal equilibrium state $\hat{\omega}^{\mathrm{can}}$ (after symmetrization of ω if necessary). The number of thermodynamical phases of the enlarged observable algebra is equal to or greater than the number of G-symmetric thermodynamical phases of the original observable algebra.

Chapter 5

The CAR Algebra

5.1 The CAR Dynamical System

We now consider a particular C^*-algebra $\mathcal{A}$, namely the CAR algebra over a Hilbert space $\mathcal{H}$. This algebra is the foundation of fermionic quantum systems. In particular, the anti-commutation relation between the annihilation and creation operators models the Pauli exclusion principle mathematically. Recall that $\mathrm{CAR}(\mathcal{H})$ is the unique and simple C^*-algebra generated by a unit $\mathbb{1}$ and elements $a(\varphi)$, $\varphi \in \mathcal{H}$, subject to the relations

$$\begin{aligned} \mathcal{H} \ni \varphi \mapsto a^*(\varphi) &\text{ is linear,} \\ \{a^*(\varphi), a(\psi)\} &= \langle \psi, \varphi \rangle \cdot \mathbb{1}, \quad \varphi, \psi \in \mathcal{H}, \\ \{a(\varphi), a(\psi)\} &= 0 = \{a^*(\varphi), a^*(\psi)\}, \quad \varphi, \psi \in \mathcal{H}, \end{aligned}$$

where $\{\,\cdot\,,\,\cdot\,\}$ denotes the anti-commutator [BR97, Thm. 5.2.5]. The C^*-norm on $\mathrm{CAR}(\mathcal{H})$ satisfies $\|a(\varphi)\| = \|\varphi\|$ for all $\varphi \in \mathcal{H}$. In [SGL24], the $\mathbb{R}$-linear field operators generating $\mathrm{CAR}(\mathcal{H})$ are introduced. We will however not take this viewpoint here. We remark that $\mathrm{CAR}(\mathcal{H})$ can be defined concretely in a representation over $\mathcal{F}_-(\mathcal{H})$, see [BR97]. This defining representation is the reason for the nomenclature of creation operators $a^*(\varphi)$ and annihilation operators $a(\varphi)$.

The dynamics and twist are given by Bogoliubov automorphisms in the setting that we will consider here. That is, we consider a strongly continuous unitary one-parameter group $U(t) = e^{itH}$ on $\mathcal{H}$ and a unitary representation $V : G \to \mathcal{U}(\mathcal{H})$ of a finite group G. The unique dynamics and twist on $\mathrm{CAR}(\mathcal{H})$ are then given by

$$\alpha_t(a(\varphi)) := a(U(t)\varphi), \qquad \gamma_s(a(\varphi)) := a(V_s\varphi), \qquad \varphi \in \mathcal{H}.$$

We furthermore assume that U and V commute pointwise, so that we obtain a finitely-twisted C^*-dynamical system denoted by $(\mathrm{CAR}(\mathcal{H}), \alpha, \gamma)$ or $\mathrm{CAR}(\mathcal{H}, U, V)$ in the sense of Definition 2.0.1. Note that $\mathrm{CAR}(\mathcal{H}, U, V)$ is concretely represented on the Fermionic Fock space $\mathcal{F}_-(\mathcal{H})$ over $\mathcal{H}$, with the dynamics and twist implemented as

$$\alpha_t(A) = \Gamma(U(t))A\Gamma(U(t))^*, \qquad \gamma_s(A) = \Gamma(V_s)A\Gamma(V_s)^*, \qquad A \in \mathrm{CAR}(\mathcal{H}).$$

Here $\Gamma(V)$ denotes the second quantization of a unitary V on $\mathcal{H}$, namely the restriction of $\bigoplus_n V^{\otimes n}$ to $\mathcal{F}_-(\mathcal{H})$.

In the following, we want to apply the general results from Chapter 4 to $\mathrm{CAR}(\mathcal{H}, U, V)$. We begin by introducing the self-dual CAR algebra $\mathrm{CAR}_{\mathrm{SD}}$, which puts the creation and annihilation operators on equal footing and thus reduces the combinatorics [Ara68; Ara71]. This is helpful for understanding the existence and uniqueness results on KMS states of the CAR algebra. To go over from the usual CAR algebra to the self-dual version, one first doubles the Hilbert space and defines the anti-unitary involution

$$\mathcal{K} = \mathcal{H} \oplus \overline{\mathcal{H}}, \quad \Gamma : \mathcal{K} \to \mathcal{K},\ (\varphi, \overline{\psi}) \mapsto (\psi, \overline{\varphi}),$$

where $\overline{\mathcal{H}}$ denotes the conjugate Hilbert space of $\mathcal{H}$. Note that Γ denotes the anti-unitary involution instead of the second quantization from now on. The one particle dynamics and G-action is then carried over to $\mathcal{K}$ by setting

$$U_{\mathrm{SD}}(t) = U(t) \oplus \overline{U(t)} \quad \text{and} \quad V_{\mathrm{SD}}(s) = V_s \oplus \overline{V_s},$$

where the generator of U_{SD} is then given by

$$H_{\mathrm{SD}} = -i \left.\frac{\mathrm{d}U_{\mathrm{SD}}}{\mathrm{d}t}\right|_{t=0} = H \oplus \overline{-H}.$$

Therefore, the dynamics and G-action commute with the anti-unitary involution Γ, whereas the generator H_{SD} anti-commutes with the involution.

The algebra of self-dual fields $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ is then the unique unital C^*-algebra generated by the self-dual fields $\Phi(\xi)$ subject to the relations

$$\begin{aligned} &\xi \in \mathcal{K} \mapsto \Phi(\xi) \text{ is } \mathbb{C}\text{-linear}; \\ &\Phi^*(\xi) := \Phi(\xi)^* = \Phi(\Gamma\xi), \quad \xi \in \mathcal{K}; \\ &\{\Phi^*(\xi), \Phi(\eta)\} = \langle \xi, \eta \rangle \mathbb{1}, \quad \xi, \eta \in \mathcal{K}. \end{aligned} \tag{5.1.1}$$

Analogously to the CAR case, the dynamics on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ is carried over from the unitary one-parameter group on $\mathcal{K}$ by

$$\alpha_t^{\mathrm{SD}}(\Phi(\xi)) = \Phi(U_{\mathrm{SD}}(t)\xi), \quad \xi \in \mathcal{K},$$

and the twist is similarly carried over as

$$\gamma_s^{\mathrm{SD}}(\Phi(\xi)) = \Phi(V_{\mathrm{SD}}(s)\xi), \quad \xi \in \mathcal{K}.$$

The creation and annihilation operators can be recovered from the self-dual fields by

$$\Phi(\varphi, 0) \mapsto a^*(\varphi), \quad \Phi(0, \overline{\varphi}) \mapsto a(\varphi), \quad \varphi \in \mathcal{H}$$

and vice versa. It is also possible to change perspective and define the self-dual CAR algebra $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ abstractly over a Hilbert space with involution $(\mathcal{K}, \Gamma)$ by the relations (5.1.1). For the finitely-twisted and dynamical version, the data are a Hilbert space with involution $(\mathcal{K}, \Gamma)$ and two unitary representations (U, V), which commute with Γ and among each other. We call such a quadruple $(\mathcal{K}, \Gamma, U, V)$ a *finitely-twisted one particle space*. Then, by choosing as the basis projection the spectral projection of H_{SD} to the positive part $P = E(\mathbb{R}_+)$, the corresponding Fock representation will be the vacuum CAR algebra from above, provided that H_{SD} does not have zero eigenmodes. The treatment in the case of zero eigenmodes is more complicated as the eigenspace might have odd dimension [Ara71]. Recall that a basis projection of $(\mathcal{K}, \Gamma)$ is an orthogonal projection $P \in \mathcal{B}(\mathcal{K})$ satisfying

$$\Gamma P \Gamma = 1 - P,$$

which only exists in case $\mathcal{K}$ is even or infinite dimensional.

To begin with, we recall the definition of a quasifree functionas on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ and in particular discuss the KMS state of $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$. As a domain for potentially unbounded functionals, we define $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)_0 \subset \mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ as the *-algebra generated by the field operators $\Phi(\xi), \xi \in \mathcal{K}$.

A functional μ on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)_0$ is called *quasifree* if $\mu(\mathbb{1}) = 1$,

$$\mu(\Phi(\xi_1) \cdots \Phi(\xi_{2n+1})) = 0,$$
$$\mu(\Phi(\xi_1) \cdots \Phi(\xi_{2n})) = (-1)^{\frac{(n-1)n}{2}} \sum_{\sigma \in \mathrm{P}_{2n}} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} \mu(\Phi(\xi_{\sigma(j)}) \Phi(\xi_{\sigma(j+n)})),$$

for all $n \in \mathbb{N}$, $\xi_1, \ldots, \xi_n \in \mathcal{H}$, where

$$\mathrm{P}_{2n} = \{\sigma \in S_{2n} \,|\, \sigma(1) < \sigma(2) < \cdots < \sigma(n) \quad \text{and} \quad \sigma(j) < \sigma(j+n) \, \forall \, 1 \leq j \leq n\}$$

is the set of all partitions of $\{1, \ldots, 2n\}$ into pairs (pairings) [Ara71]. Clearly, a quasifree functional is completely determined by its two point function.

When μ is continuous, we denote its extension to the C^*-algebra $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ by the same symbol and call it a quasifree functional on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$. This is in particular the case when μ is positive.

Consider a quasifree state μ on the self-dual CAR algebra and define the corresponding sesquilinear form

$$\mathcal{K} \times \mathcal{K} \to \mathbb{C}, \; (\xi, \eta) \mapsto \mu(\Phi^*(\xi)\Phi(\eta)).$$

It then follows by the Lax-Milgram theorem that there exists a unique positive operator $C_\mu \in \mathcal{B}(\mathcal{K})$ s.t.

$$\mu(\Phi^*(\xi)\Phi(\eta)) = \langle \xi, C_\mu \eta \rangle,$$
$$0 \leq C_\mu = C_\mu^* \leq \mathbb{1}, \tag{5.1.2}$$
$$\Gamma C_\mu^* \Gamma + C_\mu = \mathbb{1}. \tag{5.1.3}$$

We call a (possibly unbounded) operator C on $\mathcal{K}$ satisfying equation (5.1.3) a *covariance operator* and an *bounded covariance operator* if it is further bounded. If C additionally satisfies equation (5.1.2), then it is called an *positive covariance operator*.

Conversely, a positive covariance operator C induces a unique quasifree state μ_C. Therefore, the quasifree states are in one-to-one correspondence with positive covariance operators [Ara71]. By dropping the positivity of the covariance operator, one derives the following.

Proposition 5.1.1.

1) *The set of all quasifree states μ on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ is in bijection with the set of all positive covariance operators $C \in \mathcal{B}(\mathcal{K})$. The state μ_C corresponding to C has the two-point function*

$$\mu_C(\Phi(\xi)\Phi(\eta)) = \langle \Gamma\xi, C\eta \rangle. \tag{5.1.4}$$

2) *Given any bounded covariance operator $C \in \mathcal{B}(\mathcal{K})$, there exists a unique quasifree functional μ_C on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)_0$ which satisfies* (5.1.4) *and* $\mu_C(\mathbb{1}) = 1$.

3) *If the bounded covariance operator $C \in \mathcal{B}(\mathcal{K})$ is moreover selfadjoint, then the corresponding quasifree functional μ_C is hermitian.*

4) *Given any bounded covariance operator $C \in \mathcal{B}(\mathcal{K})$, the functional μ_C is invariant under γ^{SD} if, and only if, $[V_{\mathrm{SD}}(s), C] = 0$ for all $s \in G$.*

Proof. 1): It is well known (see, e.g., [BMV68; DG22]).

2): Let C be given and define $\mu_C : \mathrm{CAR}(\mathcal{H})_0 \to \mathbb{C}$ by (5.1.4), $\mu_C(\mathbb{1}) = 1$, and requiring it to be quasifree. This is a consistent definition resulting in a hermitian functional because the two-point function (5.1.4) satisfies

$$\mu_C(\{\Phi(\xi), \Phi(\eta)\}) = \langle \Gamma\xi, C\eta \rangle + \langle \Gamma\eta, C\xi \rangle = \langle \Gamma\xi, (C + \Gamma C^*\Gamma)\eta \rangle = \langle \Gamma\xi, \eta \rangle.$$

On the other hand, C is uniquely determined by μ_C.

3): If C is selfadjoint, then

$$\overline{\mu_C(\Phi(\xi)\Phi(\eta))} = \overline{\langle \Gamma\xi, C\eta\rangle} = \langle \Gamma^2\eta, C\Gamma\xi\rangle = \mu_C((\Phi(\xi)\Phi(\eta))^*).$$

4): It is clear for the two-point function. The statement then follows from the combinatorical structure of the n-point function. □

Note that the proof of item 3) shows that $\mu_C^* = \mu_{C^*}$. We further remark that the two-point function of a bounded quasifree functional μ is bounded and thus the associated covariance operator C_μ is bounded. The converse is however not true.

The following result can be found in [BR97; DG22] for the CAR-algebra and in [Ara71] for the corresponding version on the self-dual CAR algebra.

Proposition 5.1.2. *Let $(\mathcal{K}, \Gamma, U)$ be a one particle space and assume that the zero eigenspace of H_{SD} is zero, even or infinite dimensional. For every inverse temperature β there exists a unique KMS state ω_β on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$, namely*

$$\omega_\beta = \mu_{C_\beta}, \qquad C_\beta = (\mathbb{1} + e^{-\beta H_{\mathrm{SD}}})^{-1}.$$

If $V_{\mathrm{SD}} \in \mathcal{U}(\mathcal{K})$ commutes with Γ and U, then ω_β is γ^{SD}-invariant, where

$$\gamma^{\mathrm{SD}}(\Phi(\xi)) = \Phi(V_{\mathrm{SD}}\xi).$$

Note that in the tracial case $H_{\mathrm{SD}} = 0$ this describes the unique tracial state μ_0 on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$. In case $\ker H_{\mathrm{SD}}$ is odd dimensional, the question is more subtle and thus not covered here. The resulting GNS representation might not be a factor and thus might lead to multiple KMS states as they are related to the positive central elements [Ara71]. This does however not concern us as we are mainly interested in the case $\ker H_{\mathrm{SD}} = \{0\}$. In the case of $\mathrm{CAR}(\mathcal{H}, U)$, the corresponding self-dual Hamiltonian is given by $H_{\mathrm{SD}} = H \oplus \overline{-H}$ and therefore its space of zero modes is zero, even or infinite dimensional.

5.2 KMS States on the CAR Algebra

We now investigate the extension problem for the state ω_β described in Proposition 5.1.2 to the crossed product of $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$ by the twist γ. In the following we drop the index SD for U_{SD}, H_{SD} and V_{SD}, since we will mostly deal with the self-dual version from now on.

To describe this situation, we note that as a consequence of $U(t) = e^{itH}$ commuting with V, the spectral measure E of H commutes with the unitary G-action. Moreover, as G is finite, the unitary operators V_s are all of finite order, i.e. $V_s^N = \mathbb{1}$ for some $N \in \mathbb{N}$.

In the following, we call an operator $V \in \mathcal{U}(\mathcal{K})$ commuting with Γ and E a *one particle twist* and a *finite order one particle twist* if it additionally is of finite order. For a finite order one particle twist $V \in \mathcal{U}(\mathcal{K})$, the spectrum $\sigma(V)$ is discrete and finite. Consequently, 1 is either an eigenvalue or not contained in the spectrum. We thus consider the decomposition $\mathcal{K} = \mathrm{Eig}_1(V) \oplus \mathrm{Eig}_1(V)^\perp$ and denote $\mathcal{K}_\perp := (\mathrm{Eig}_1(V))^\perp$. By the V-odd part of the Hamiltonian H we mean the restriction $H|_{\mathcal{K}_\perp}$ and denote it by $H_\perp$.

According to Theorem 4.3.1, the first step to understanding the KMS states on this C^*-algebra is to determine the γ-twisted KMS functionals ρ_β of $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$. We consider the case without zero modes, i.e. $\ker H = \{0\}$, and start with a (finite order) one particle twist V. After understanding the twisted functionals for a single automorphism, will consider the entire G-action. We mention in passing that the twisted tracial functionals for $G = \mathbb{Z}_2$ are discussed in [SGL24].

As mentioned before, the twist γ is implemented by $\Gamma(V)$ on $\mathcal{F}_-(\mathcal{H})$, where Γ denotes second quantization. The operator $\Gamma(V_s)$ certainly lies in $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ if $\mathrm{Eig}_1(V_s)^\perp$ is finite-dimensional. This can be seen by going to a U-invariant eigenbasis $\varphi_1, \ldots, \varphi_n$ of V_s on $\mathrm{Eig}_1(V_s)^\perp$ with eigenvalues $\mu_k = e^{i\nu_k}$ and realizing that the finite product

$$\mathcal{V}_s := \prod_{k=1}^{n} e^{i\nu_k \Phi(\varphi_k)\Phi(\Gamma\varphi_k)}$$

implements γ_s in $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$. Thus, a γ_s-twisted KMS functional can be defined as in Lemma 4.3.2. It is however not clear if this family is G-equivariant and positively compatible at this point.

Twisted KMS functionals can well be unbounded, see [BG07; Hil15] for results motivated by supersymmetry. We are however aiming for twisted KMS functionals that are continuous because they are dominated by ω_β. We therefore restrict already here to those functionals ρ_β that at least have bounded n-point functions, i.e. for any $n \in \mathbb{N}$, there exists $c_n > 0$ such that

$$|\rho_\beta(\Phi(\xi_1) \cdots \Phi(\xi_n))| \leq c_n \|\xi_1\| \cdots \|\xi_n\|.$$

Proposition 5.2.1. *Let $(\mathcal{K}, \Gamma, U)$ be a one particle space, $V \in \mathcal{U}(\mathcal{K})$ a one particle twist and* $\ker H = \{0\}$*. Let ρ_β be a γ-twisted KMS functional of* $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$ *with bounded n-point functions.*

1) *If $-1 \in \sigma(Ve^{-\beta H})$, then $\rho_\beta = 0$.*
2) *If $-1 \notin \sigma(Ve^{-\beta H})$, then*

$$\rho_\beta|_{\mathrm{CAR}(\mathcal{H})_0} = \rho_\beta(\mathbb{1}) \cdot \mu_{C_\beta^V}, \qquad C_\beta^V = (\mathbb{1} + Ve^{-\beta H})^{-1}.$$

In particular, ρ_β is uniquely determined by its value $\rho_\beta(\mathbb{1})$.

Proof. To improve readability of our formulae, we will sometimes use subscript notation $\Phi_\xi = \Phi(\xi)$ in this proof.

We begin by calculating the one- and two-point function of ρ_β. For η entire analytic for the one-parameter group U (and hence Φ_η^* entire analytic for α), the γ-twisted KMS condition requires $\rho_\beta(\Phi(e^{-\beta H}\eta)) = \rho_\beta(\mathbb{1} \cdot \Phi(e^{-\beta H}\eta)) = \rho_\beta(\Phi(\eta))$ and hence $\rho_\beta(\Phi((1 - e^{-\beta H})\eta)) = 0$. Since H is injective and ρ_β has a bounded one-point function, this implies that $\rho_\beta(\Phi(\psi)) = 0$ for all $\psi \in \mathcal{K}$.

For the two-point function, the γ-twisted KMS condition and γ-invariance of ρ_β yield

$$\begin{aligned}\rho_\beta(\Phi(\xi)\Phi(e^{-\beta H}\eta)) &= \rho_\beta(\Phi(\xi)\alpha_{i\beta}(\Phi(\eta))) = \rho_\beta(\Phi(\eta)\Phi(V\xi)) \\ &= -\rho_\beta(\Phi(\xi)\Phi(V^*\eta)) + \rho_\beta(\langle\Gamma\eta, V\xi\rangle\mathbb{1}).\end{aligned}$$

Combining these equations with $\Gamma V = V\Gamma$ gives

$$\rho_\beta(\Phi(\xi)\Phi((e^{-\beta H} + V^*)\eta)) = \rho_\beta(\mathbb{1}) \cdot \langle\Gamma\xi, V^*\eta\rangle.$$

This shows that $\rho_\beta(\Phi(\xi)\Phi(\eta)) = \rho_\beta(\mathbb{1}) \cdot \langle\Gamma\xi, C_\beta^V\eta\rangle$ for analytic vectors ξ, η, provided that

$$C_\beta^V := (\mathbb{1} + Ve^{-\beta H})^{-1}$$

exists and $\eta \in \mathrm{dom}(C_\beta^V)$.

We now show that $(\mathbb{1} + Ve^{-\beta H})$ is invertible. For this take $\varphi \in \ker(\mathbb{1} + Ve^{-\beta H})$ with $\|\varphi\| = 1$ and note that this in particular implies $\varphi \in \mathrm{dom}(e^{-\beta H})$. Due to V and H commuting, we find

$$e^{-\beta H}\varphi = -V^*\varphi \in \mathrm{dom}(e^{-\beta H}).$$

Hence, φ is in the domain of $e^{-n\beta H}$ for all $n \in \mathbb{N}$ by induction. Regarding the norm it moreover holds $\left\|e^{-\beta H}\varphi\right\| = \|-V^*\varphi\| = \|\varphi\| = 1$ and hence

$$1 = \left\|e^{-n\beta H}\varphi\right\|^2 = \int_{\sigma(H)} e^{-2n\beta\lambda} d\mu_\varphi(\lambda) \tag{5.2.1}$$

follows inductively in combination with the spectral theorem. Assuming that the support of μ_φ is not $\{0\}$, i.e. $\mu_\varphi \neq \delta_0$, there exists a point $p \in \sigma(H)$ and an ϵ-neighborhood that does not contain 0. Then the right-hand side of equation (5.2.1) would diverge for either $n \to \pm\infty$. Thus $\mu_\varphi = \delta_0$ and $\varphi \in \ker(H)$ in contradiction to the assumption.

As ρ_β has bounded two-point function, we conclude

$$\rho_\beta(\Phi(\xi)\Phi(\eta)) = \rho_\beta(\mathbb{1}) \cdot \langle\Gamma\xi, C_\beta^V\eta\rangle$$

for all $\xi, \eta \in \mathcal{K}$ by continuity. Note that C_β^V is unbounded if, and only if, $-1 \in \sigma(Ve^{-\beta H})$. Since ρ_β has a bounded two-point function, we conclude $\rho_\beta(\mathbb{1}) = 0$ in this case. Moreover, C_β^V satisfies the covariance equation $C_\beta^V + \Gamma(C_\beta^V)^*\Gamma = \mathbb{1}$.

To complete the proof, we only need to show that ρ_β is, up to the prefactor $\rho_\beta(\mathbb{1})$, quasifree. This argument proceeds along the same lines as in the untwisted case. The assertion trivially holds for zero fields, which starts the induction. Let $n \in \mathbb{N}$ and assume that the assertion holds for $2n$ fields. We will now do the induction step $2n \to 2n+k$, where we consider the odd case $k=1$ and even case $k=2$ at the same time to simplify the combinatorics. Take $\xi_i \in \mathcal{K}$ entire analytic for H.

$$\begin{aligned}\rho_\beta(\Phi(e^{+\beta H}\xi_1)\cdots\Phi_{\xi_{2n+k}}) &= \rho_\beta(\Phi_{\xi_2}\cdots\Phi_{\xi_{2n+k}}\Phi(V\xi_1)) \\ &= (-1)^{2n+k-1}\rho_\beta(\Phi(V\xi_1)\cdots\Phi_{\xi_{2n+k}}) \\ &+ \sum_{j=2}^{2n+k}(-1)^{2n+k-j}\langle\Gamma V\xi_1,\xi_j\rangle\cdot\rho_\beta(\Phi_{\xi_2}\cdots\widehat{\Phi}_{\xi_j}\cdots\Phi_{\xi_{2n+k}}),\end{aligned}$$

where the last equality was derived by using the anti-commutation relations of the self-dual fields and $\widehat{\Phi}_{\xi_j}$ denotes that the j-th field gets omitted. The above equation can be rewritten as

$$\begin{aligned}&\rho_\beta(\Phi((e^{+\beta H}+(-1)^kV)\xi_1)\cdots\Phi_{\xi_{2n+k}}) \\ &= \sum_{j=2}^{2n+k}(-1)^{k+j}\langle\Gamma V\xi_1,\xi_j\rangle\cdot\rho_\beta(\Phi_{\xi_2}\cdots\widehat{\Phi}_{\xi_j}\cdots\Phi_{\xi_{2n+k}}).\end{aligned}$$

Similar to the above discussion, $e^{+\beta H}+(-1)^kV$ is invertible. Incorporating the factor $(-1)^k$ into the covariance operator and using $\Gamma H = -H\Gamma$ shows

$$\rho_\beta(\Phi_{\xi_1}\cdots\Phi_{\xi_{2n+k}}) = \sum_{j=2}^{2n+k}(-1)^j\langle\Gamma\xi_1,C_\beta^{(-1)^kV}\xi_j\rangle\cdot\rho_\beta(\Phi_{\xi_2}\cdots\widehat{\Phi}_{\xi_j}\cdots\Phi_{\xi_{2n+k}}).$$

As the n-point functions are bounded, we can drop the assumption of analyticity of the vectors ξ_i. Note that for the odd case $k=1$, the n-point functions are reduced to one-point functions and thus vanish. The even case $k=2$ is now the recursion relation for the quasifree functional. The proof of this can be found in the appendix A. □

The crucial question is now whether the twisted KMS functionals are dominated by the unique KMS state discussed before, as this is a necessary criterion for the extension problem. In particular, we have not yet clarified under which conditions $\mu_{C_\beta^V}$ is continuous, which is a consequence of $\rho_\beta(\mathbb{1}) \neq 0$ and ρ_β being dominated by ω_β.

For a given Hamiltonian βH and a finite order one particle twist V, we define the following:

$$c_{\beta H,V} := \prod_{k\in\mathbb{N}}\frac{\left|1+\mu_k e^{-\beta\lambda_k}\right|}{1+e^{-\beta\lambda_k}},$$

where the $(\beta\lambda_k)_{k\in\mathbb{N}}$ is the sequence of positive spectral values of $\beta H_\perp$ counted with multiplicity and $(\mu_k)_{k\in\mathbb{N}}$ is the corresponding sequence of eigenvalues of V, i.e. $(\beta\lambda_k, \mu_k)_{k\in\mathbb{N}}$ runs over the joint spectrum of $\beta H_\perp$ and V on $E^{\beta H}(\mathbb{R}_+)\mathcal{K}_\perp$. If $\beta H_\perp$ does not have pure point spectrum, we set $c_{\beta H,V} = 0$ as the above product over all spectral values would vanish.

Proposition 5.2.2. *Let* $(\mathcal{K}, \Gamma, U)$ *be a one particle space,* $V \in \mathcal{U}(\mathcal{K})$ *a finite order one particle twist s.t.* $\dim \mathrm{Eig}_1(V)^\perp = \infty$ *and* $\ker H = \{0\}$.

The following are equivalent:

1) *There exists a non-vanishing* γ*-twisted KMS functional of* $\mathrm{CAR_{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$ *dominated by* ω_β *(see Prop. 5.2.1);*

2) $c_{\beta H,V} > 0$;

3) $\mathrm{Tr}_{\mathrm{Eig}_1(V)^\perp}\left(e^{-|\beta H_\perp|}\right) < \infty$.

In this case, $\mu_{C_\beta^V}$ *is bounded and*

$$\begin{aligned}&\{\rho_\beta \in \mathcal{F}_\beta(\mathrm{CAR_{SD}}(\mathcal{K}, \Gamma, \mathrm{U}), \gamma) \,|\, \rho_\beta \text{ dominated by } \omega_\beta\} \\ &= \{c \cdot \mu_{C_\beta^V} \,:\, |c| \le c_{\beta H,V}\}.\end{aligned} \tag{5.2.2}$$

Proof. 1) $\Rightarrow$ 2): Let ρ_β be a non-vanishing γ-twisted KMS functional dominated by ω_β. We now choose a sequence of vectors $\xi_1, \dots, \xi_n \in \mathrm{Eig}_1(V)^\perp$ adapted to the one particle structure. Take ξ_i mutually orthogonal unit vectors, such that $P\xi_i = \xi_i$ for the basis projection $P = E^{\beta H}((0, \infty))$, which will simplify the estimate and

$$\langle \xi_j, (\mathbb{1} + e^{-\beta H})^{-1}\xi_k\rangle = 0 = \langle \xi_j, (\mathbb{1} + Ve^{-\beta H})^{-1}\xi_k\rangle, \quad \forall j \neq k.$$

Such vectors exist for any $n \in \mathbb{N}$ because of $\dim(\mathrm{Eig}_1(V)^\perp) = \infty$. Moreover, $\langle \Gamma\xi_k, \xi_j\rangle = 0$ as P is a basis projection. In view of the quasifree structure of ω_β and ρ_β, the operator $A := \Phi(\xi_1)\cdots\Phi(\xi_n)$ satisfies

$$\rho_\beta(A^*A) = \rho_\beta(\mathbb{1}) \prod_{k=1}^n \rho_\beta(\Phi(\xi_k)^*\Phi(\xi_k)) = \rho_\beta(\mathbb{1}) \prod_{k=1}^n \langle \xi_k, (\mathbb{1} + Ve^{-\beta PHP})^{-1}\xi_k\rangle,$$

$$\rho_\beta(AA^*) = \rho_\beta(\mathbb{1}) \prod_{k=1}^n \langle \Gamma\xi_k, (\mathbb{1} + Ve^{-\beta H})^{-1}\Gamma\xi_k\rangle = \rho_\beta(\mathbb{1}) \prod_{k=1}^n \langle \xi_k, (\mathbb{1} + V^*e^{+\beta PHP})^{-1}\xi_k\rangle,$$

$$\omega_\beta(A^*A) = \prod_{k=1}^n \omega_\beta(\Phi(\xi_k)^*\Phi(\xi_k)) = \prod_{k=1}^n \langle \xi_k, (\mathbb{1} + e^{-\beta PHP})^{-1}\xi_k\rangle,$$

$$\omega_\beta(AA^*) = \prod_{k=1}^n \langle \xi_k, (\mathbb{1} + e^{+\beta PHP})^{-1}\xi_k\rangle.$$

Note that this is written in terms of the positive operator PHP. As ω_β dominates ρ_β, we have $|\rho_\beta(A^*A)| \le \omega_\beta(A^*A)$ and $|\rho_\beta(AA^*)| \le \omega_\beta(AA^*)$, i.e.

$$|\rho_\beta(\mathbb{1})| \le \prod_{k=1}^n \frac{\langle \xi_k, (\mathbb{1} + e^{\pm\beta PHP})^{-1}\xi_k\rangle}{|\langle \xi_k, (\mathbb{1} + V^{\mp 1}e^{\pm\beta PHP})^{-1}\xi_k\rangle|}, \tag{5.2.3}$$

provided $\left|\langle\xi_k, (\mathbb{1} + V^{\mp 1}e^{\pm\beta PHP})^{-1}\xi_k\rangle\right| \neq 0$.

These inequalities hold for both signs $\pm$ and can be used to obtain spectral information about the Hamiltonian βH. Note however that the inequalities that would be derived from $\rho_\beta((AA^*)^*(A^*A))$ do not give any new information due to the anti-commutation relations of the field operators. Thus, it is only meaningful to take the sequence ξ_i in one half of the spectrum, as done above. We denote the spectral projections by $E^{\beta H}$. Suppose that there exists $r > 0$ such that $\mathcal{K}^\perp_{r,+} := E^{\beta H}([0, r])(\mathrm{Eig}_1(V)^\perp)$ is infinite-dimensional. Then, for any $n \in \mathbb{N}$, we may choose $\xi_1, \ldots, \xi_n \in \mathcal{K}^\perp_{r,+}$ as eigenvectors of V with eigenvalues $\mu_k \in S_1\backslash\{1\}$ as V is of finite order. One then obtains the estimate

$$\begin{aligned} I_k &:= \left|\langle\xi_k, (1 + Ve^{-\beta PHP})^{-1}\xi_k\rangle\right| = \int_{[0,r]} \left|1 + \mu_k e^{-\lambda}\right|^{-1} d\mu_{\xi_k}(\lambda) \\ &> \int_{[0,r]} (1 + e^{-\lambda})^{-1} d\mu_{\xi_k}(\lambda) \end{aligned}$$

where it was used that $\left|1 + \mu_k e^{-\lambda}\right| = \left|\overline{\mu_k} + e^{-\lambda}\right| < |\mu_k| + e^{-\lambda} = 1 + e^{-\lambda}$ since $\mu_k \neq 1$. We compare this with the numerator

$$I := \langle\xi_k, (1 + e^{-\beta PHP})^{-1}\xi_k\rangle = \int_{[0,r]} (1 + e^{-\lambda})^{-1} d\mu_{\xi_k}(\lambda)$$

of equation (5.2.3) and define $c_k := \frac{I}{I_k} < 1$. This shows the following limit

$$\rho_\beta(\mathbb{1}) \leq \prod_{k=1}^n \frac{I}{I_k} = \prod_{k=1}^n c_k \to 0,$$

where it is used that only finitely many different μ_k arise. We thus conclude that $\rho_\beta(\mathbb{1}) = 0$.

As $\rho_\beta(\mathbb{1}) = 0$ implies the contradiction $\rho_\beta = 0$, we see by symmetry of $\sigma(H_\perp)$ that $E^{\beta H}([-r, r])(\mathrm{Eig}_1(V)^\perp)$ is finite-dimensional for all $r > 0$. This argument shows already that $\sigma(H_\perp)$ consists only of eigenvalues with finite multiplicity and no finite accumulation point. Let us denote the eigenvalues $(\beta\lambda_k)_{k\in\mathbb{N}}$ (repeated according to multiplicity) and choose the ξ_k to be normalized corresponding eigenvectors of $\beta H_\perp$. As V and H commute and the eigenspaces of H are finite-dimensional, we can diagonalize V on each eigenspace and choose ξ_k as an eigenvector of V with eigenvalue μ_k as well. Then the inequality in (5.2.3) yields

$$0 < |\rho_\beta(\mathbb{1})| \leq \prod_{k=1}^n \left|\frac{1 + \mu_k^{\mp 1}e^{\pm\beta\lambda_k}}{1 + e^{\pm\beta\lambda_k}}\right| = \prod_{k=1}^n \frac{\left|1 + \mu_k^{\mp 1}e^{\pm\beta\lambda_k}\right|}{1 + e^{\pm\beta\lambda_k}} = \prod_{k=1}^n \frac{\left|1 + \mu_k e^{-\beta\lambda_k}\right|}{1 + e^{-\beta\lambda_k}} \quad (5.2.4)$$

for all $n \in \mathbb{N}$. The symmetry of $\sigma(\beta H_\perp)$ assures that the inequalities for $+$ and $-$ are the same. Taking the limit $n \to \infty$ shows

$$0 \neq \prod_{k\in\mathbb{N}} \frac{\left|1 + \mu_k e^{-\beta\lambda_k}\right|}{1 + e^{-\beta\lambda_k}} = \prod_{k\in\mathbb{N}} \frac{\left|1 + \overline{\mu_k} e^{+\beta\lambda_k}\right|}{1 + e^{+\beta\lambda_k}},$$

where the $(\beta\lambda_k)_{k\in\mathbb{N}}$ is the sequence of positive spectral values of βH counted with multiplicity and $(\mu_k)_{k\in\mathbb{N}}$ is the corresponding sequence of eigenvalues of V. Hence $c_{\beta H,V} > 0$.

2) $\Leftrightarrow$ 3): By standard estimates translating infinite products into infinite sums, one obtains

$$c_{\beta H,V} \neq 0 \Leftrightarrow \sum_{k\in\mathbb{N}} \left(1 - \frac{\left|1+\mu_k e^{-\beta\lambda_k}\right|}{1+e^{-\beta\lambda_k}}\right) < \infty.$$

This series can be compared to $\mathrm{Tr}_{\mathrm{Eig}_1(V)^\perp}\left(e^{-|\beta H_\perp|}\right)$. We denote $\mu_k = e^{i\nu_k}$ and make the comparison in the limit $\beta\lambda_k \to +\infty$.

$$\frac{1 - \frac{\left|1+\mu_k e^{-\beta\lambda_k}\right|}{1+e^{-\beta\lambda_k}}}{e^{-|\beta\lambda_k|}} = \frac{2(1-\cos(\nu_k))}{(1+e^{-\beta\lambda_k})(1+e^{-\beta\lambda_k}+\sqrt{\sin(\nu_k)^2 + (\cos(\nu_k)+e^{-\beta\lambda_k})^2})}$$

This is bounded in the limit by $\liminf = 1-\cos(\nu_{\min}) > 0$ and $\limsup = 2$, where $\mu_{\min} = e^{i\nu_{\min}}$ is the eigenvalue of V which is closest to 1 on $\mathrm{Eig}_1(V)^\perp$. By the comparison criterion, $c_{\beta H,V} \neq 0$ is equivalent to $\mathrm{Tr}_{\mathrm{Eig}_1(V)^\perp}\left(e^{-|\beta H_\perp|}\right) < \infty$. Here it was used that the $\mathrm{Tr}_{\mathrm{Eig}_1(V)^\perp}\left(e^{-|\beta H_\perp|}\right)$ is simply double the trace over the positive spectrum.

3) $\Rightarrow$ 1): The idea of this part of the proof is to use Proposition 4.4.7. We choose an orthonormal basis $(\xi_k)_{k\in\mathbb{N}}$ of $P(\mathrm{Eig}_1(V)^\perp)$ of $\beta H_\perp$ and V simultaneously with corresponding eigenvalues $(\beta\lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k = e^{i\nu_k})_{k\in\mathbb{N}}$. Here $P = E^{\beta H}(\mathbb{R}_+)$ the projection to the positive part of βH and note that $\beta\lambda_k > 0$ for all $k \in \mathbb{N}$. Consider the operators $P_k := \Phi(\xi_k)\Phi(\Gamma\xi_k)$. The anti-commutation relations imply that the P_k are mutually commuting orthogonal projections satisfying

$$e^{i\nu_k P_k} = \mathbb{1} + (e^{i\nu_k}-1)P_k = \Phi(V\xi_k + \Gamma\xi_k)\Phi(\xi_k + \Gamma\xi_k).$$

The unitaries implementing the weakly inner grading from Proposition 4.4.7 are here taken as

$$v_n := \prod_{k=1}^{n} e^{i\nu_k P_k},$$

as Ad_{v_n} approximates γ. As a suitable subalgebra, we take $\mathcal{A}_0$ to be the *-algebra generated by the $\Phi(\xi_k)$, $k \in \mathbb{N}$, and $\Phi(\eta)$, $\eta \in \mathrm{Eig}_1(V)$ analytic for U. This algebra satisfies the assumptions of Proposition 4.4.7. Moreover, we have

$$v_n\Phi(\xi_m)v_n^* = \begin{cases} e^{i\nu_m}\Phi(\xi_m) & m \leq n \\ \Phi(\xi_m) & m > n \end{cases}, \qquad v_n\Phi(\eta)v_n^* = \Phi(\eta), \quad \eta \in \mathrm{Eig}_1(V).$$

This yields $\|\mathrm{Ad}_{v_n}(A) - \gamma(A)\| \to 0$ for any $A \in \mathcal{A}_0$.

In view of the quasifree structure of ω_β, we have

$$\begin{aligned}\omega_\beta(v_n) &= \omega_\beta\left(\prod_{k=1}^{n}\Phi(V\xi_k+\Gamma\xi_k)\Phi(\xi_k+\Gamma\xi_k)\right)\\ &= \prod_{k=1}^{n}\omega_\beta\left(\Phi(V\xi_k+\Gamma\xi_k)\Phi(\xi_k+\Gamma\xi_k)\right)\\ &= \prod_{k=1}^{n}\left(\frac{e^{i\nu_k}}{1+e^{+\beta\lambda_k}}+\frac{1}{1+e^{-\beta\lambda_k}}\right) = \prod_{k=1}^{n}\frac{1+e^{i\nu_k}e^{-\beta\lambda_k}}{1+e^{-\beta\lambda_k}}.\end{aligned}$$

The absolute value of this product was already analyzed in 2) $\Leftrightarrow$ 3) and shown to converge to $c_{\beta H,V}\neq 0$ under the trace class assumption. It is then clear that for $n<m$, we have $\omega_\beta(v_n^*v_m)\to 1$ as $n,m\to\infty$, verifying another assumption of Proposition 4.4.7.

We may therefore apply Proposition 4.4.7 to conclude that $\rho_\beta(a) := \lim_n \omega_\beta(av_n)$ is indeed a γ-twisted KMS functional dominated by ω_β, this proves 1).

By application of Proposition 5.2.1, we also get $\rho_\beta = \rho_\beta(\mathbb{1})\,\mu_{C_\beta^V}$. According to the estimate (5.2.4), $c_{\beta H,V}$ is an upper bound for $|\rho_\beta(\mathbb{1})|$ for any γ-twisted KMS functional dominated by ω_β. Therefore this upper bound is attained, and (5.2.2) follows. □

Hillier studied boundedness questions for twisted KMS-functionals in [Hil15]. His Theorem A.4 appears to be related to some parts of our analysis above in case the twist V is selfadjoint. We mention in passing that a similar analysis is possible in case V is not of finite order. On the one hand, our analysis generalizes to $V = e^{i\frac{\mu}{2\pi}}$ with μ irrational. On the other hand, V can be contained in $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K},\Gamma)$ under certain spectral assumptions on V and thus defines a twisted functional.

We do not need a concrete realization of the GNS representation of ω_β for our analysis. For completeness, let us still point out that this representation can be realized as a double Fock representation, see [AW64; Ara71; BJL02]. For this, we work with the basis projection $P = E^H((0,\infty))$ and consider the associated splitting $\mathcal{H} = P\mathcal{K}$, $\Phi(\varphi\oplus\overline{\psi}) = a^*(\varphi)+a(\psi)$, $H = PH_{\mathrm{SD}}P$, $V = PV_{\mathrm{SD}}P$ and Fock vacuum Ω. For emphasis, the index SD is used again for the self-dual objects. Then the representation associated to the KMS state ω_β can be realized as

$$\begin{aligned}\mathcal{F}_-(\mathcal{H})_\beta = \mathcal{F}_-(\mathcal{H})\otimes\overline{\mathcal{F}_-(\mathcal{H})},\quad \Omega_\beta = \Omega\otimes\overline{\Omega},\quad V_\beta = \Gamma(V)\otimes\overline{\Gamma(V)},\\ a_\beta^*(\varphi) := \pi_\beta(a^*(\varphi)) = a^*(\sqrt{T}\varphi)\otimes\overline{1}+(-1)^N\otimes\overline{a(\sqrt{1-T}\varphi)},\\ a_\beta(\varphi) := \pi_\beta(a(\varphi)) = a(\sqrt{T}\varphi)\otimes\overline{1}+(-1)^N\otimes\overline{a^*(\sqrt{1-T}\varphi)},\end{aligned}$$

where $T = (1+e^{-\beta H})^{-1}$, Γ denotes second quantization and $\overline{\mathcal{F}_-(\mathcal{H})}$ denotes the complex conjugate Hilbert space of $\mathcal{F}_-(\mathcal{H})$. The modular conjugation of the enveloping von Neumann algebra $\mathfrak{M}$ is given by $J = [I_1 \otimes \overline{I_1}]F$, where F denotes the tensor flip and $I_1 = (-1)^{\frac{N(N-1)}{2}}$.

Physically, the thermal Fock space $\mathcal{F}_-(\mathcal{H})_\beta$ can be interpreted as describing both particles $\mathcal{F}_-(\mathcal{H})$ and thermal holes $\overline{\mathcal{F}_-(\mathcal{H})}$. The thermal creation operator $a^*_\beta(\varphi)$ is in superposition between creation of a particle and annihilation of a thermal hole, whereas the thermal annihilation operator $a_\beta(\varphi)$ is in superposition between annihilation of a particle and creation of a hole. Both operators do not annihilate the thermal vacuum Ω_β but rather create a particle resp. thermal hole. The modular conjugation switches particles with holes and thus describes a symmetry between these.

In the notation of the proof of Proposition 5.2.2, the product

$$\mathcal{V} := \prod_{k=1}^{\infty} e^{i\nu_k a^*_\beta(\varphi_k) a_\beta(\varphi_k)}$$

converges strongly to a non-zero operator if

$$\mathrm{Tr}_{\mathrm{Eig}_1(V)^\perp}\left(e^{-|\beta H_\perp|}\right) < \infty$$

and vanishes otherwise. It is then unitary and $\mathcal{V} \in \mathcal{Z}(\mathfrak{M}, \overline{\gamma})$. As the KMS state ω_β is unique, $\mathfrak{M}$ is factor and therefore, Corollary 4.4.4 can be applied. This shows $\mathcal{Z}(\mathfrak{M}, \overline{\gamma}) = \mathcal{V} \cdot \mathbb{C}$.

In applications to finite group actions, the above proposition specifies when twisted functionals corresponding to a single automorphism exist or vanish. Putting the individual functionals together works well, as long as the V_s have a common eigenbasis. This should be seen as a consequence of the G-action being given by Bogoliubov transformations.

Theorem 5.2.3. *Let $(\mathcal{K}, \Gamma, U, V)$ be a finitely-twisted one particle space and* $\ker H = \{0\}$. *Further assume that the* $\mathrm{Eig}_1(V_s)$ *coincide for all $s \neq e$ and denote* $\mathcal{K}_\perp = \mathrm{Eig}_1(V_s)^\perp$.

If $\mathrm{Tr}_{\mathcal{K}_\perp}\left(e^{-|\beta H_\perp|}\right) < \infty$ *and either G is abelian or* $\dim \mathrm{Eig}_\lambda(H) = 1$ *for all $\lambda \in \sigma(H)$, then there exists a unitary representation $\mathcal{V} : G \to \mathfrak{M}_{\omega_\beta}$ satisfying $\mathcal{V} \in \mathcal{S}_{\overline{\gamma}}(G, \mathfrak{M}_{\omega_\beta})$ and*

$$\mathcal{S}_\beta(\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma) \rtimes \mathrm{G})_{\omega_\beta} \simeq \mathrm{Char}(G) \cdot \mathcal{V};$$
$$\partial_e(\mathcal{S}_\beta(\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma) \rtimes \mathrm{G})_{\omega_\beta}) \simeq \mathrm{IrrChar}(G) \cdot \mathcal{V}.$$

On the converse, if $\mathrm{Tr}_{\mathcal{K}_\perp}\left(e^{-|\beta H_\perp|}\right) = \infty$, *then $\hat{\omega}^{\mathrm{can}}_\beta$ is the unique KMS state on* $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma) \rtimes \mathrm{G}$.

Proof. In case $\mathrm{Tr}_{\mathcal{K}_\perp}\left(e^{-|\beta H_\perp|}\right) = \infty$, Proposition 5.2.2 tells us that there exist no γ_s-twisted KMS functionals on $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma, \mathrm{U})$ for $s \neq e$. The absence of twisted functionals directly implies that the KMS state $\hat{\omega}^{\mathrm{can}}_\beta$ is unique, see Corollary 4.5.7.

Let now the trace class condition be satisfied. By Proposition 5.2.2, for every $s \in G$ there exists a γ_s-twisted KMS functional $\rho_{\beta,s}$ dominated by ω_β in case $\mathrm{Eig}_1(V_s)^\perp$ is infinite dimensional. In case $\mathrm{Eig}_1(V_s)^\perp$ is finite dimensional, the discussion at the beginning of the chapter applies. Here $\rho_{\beta,e} = \omega_\beta$ and $\rho_{\beta,s}(\cdot) = \langle \Omega, (\cdot)\mathcal{V}_s\Omega \rangle$, where $\mathcal{Z}(\mathfrak{M}, \overline{\gamma}_s) \ni \mathcal{V}_s = \prod_{n=1}^{\infty} e^{i\nu_{s,k}P_{s,k}}$ as a SOT-limit in the notation of Proposition 5.2.2. We now show that $\mathcal{V} : G \to \mathfrak{M}$ is a unitary representation in case $\dim \mathrm{Eig}_\lambda(H) = 1$ for $\lambda \in \sigma(H)$ or G abelian.

As $V : G \to \mathcal{U}(\mathcal{K})$ is a unitary representation and commutes with the spectral measure of H, we find $V_{rs}|_{\mathrm{Eig}_\lambda(H)} = (V_r V_s)|_{\mathrm{Eig}_\lambda(H)}$ for every $\lambda \in \sigma(H)$. Using that $\dim \mathrm{Eig}_\lambda(H) = 1$ or G abelian, we find that all V_s can be diagonalized simultaneously on $P\mathcal{K}$. Therefore, on taking the simultaneous eigenbasis $(\xi_k)_{k\in\mathbb{R}}$ on $P\mathcal{K}$, we find in particular $P_{s,k} = P_{r,k} =: P_k$ is independent of the group element. Taking the diagonal sequence yields

$$\mathcal{V}_s\mathcal{V}_r = \lim_N \prod_{k=1}^{N} e^{i\nu_{s,k}P_k} \prod_{l=1}^{N} e^{i\nu_{r,l}P_l} = \lim_N \prod_{k=1}^{N} e^{i(\nu_{s,k}+\nu_{r,k})P_k} = \lim_N \prod_{k=1}^{N} e^{i\nu_{sr,k}P_k} = \mathcal{V}_{sr},$$

where it was used that $V_{rs}\xi_k = V_r V_s \xi_k = e^{i(\nu_{s,k}+\nu_{r,k})}\xi_k$. Having shown that γ is weakly inner, Corollary 4.6.6 can be utilized and shows the assertion. □

Observe that in case $1 \notin \sigma(V_s)$ for all $s \neq e$, the trace condition of Theorem 5.2.3 of the self-dual Hamiltonian βH_{SD} is equivalent to the actual Gibbs condition of $\beta H := P\beta H_{\mathrm{SD}}P$, where $P = E^{\beta H_{\mathrm{SD}}}((0,\infty))$ is the vacuum basis projection. Hence in this case, the second quantized Hamiltonian also satisfies the Gibbs condition, namely $\mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})}(e^{-\beta d\Gamma(H)}) < \infty$ [BR97, Prop. 5.2.22]. We can therefore use the same idea as in Lemma 4.1.4 and obtain a Gibbs extension $\hat{\omega}_\beta^{\mathrm{Gibbs}}$ of ω_β to the crossed product. Since

$$\hat{\omega}_\beta^{\mathrm{Gibbs}}(V_s) = \prod_{k=1}^{\infty} \frac{1 + e^{i\nu_k}e^{-\beta\lambda_k}}{1 + e^{-\beta\lambda_k}} = \hat{\omega}_\beta^{\mathcal{V}}((-1)^N),$$

this extension coincides with the KMS $\hat{\omega}_\beta^{\mathcal{V}}$ defined by the unitary representation $\mathcal{V}$.

This concludes our analysis of the KMS states of the crossed product for the CAR system. We have not discussed the most general case of non-trivial kernel $\ker H \neq \{0\}$ which mixes the tracial case $H = 0$ with the case $\ker H = \{0\}$. We expect that this can be done efficiently by splitting $\mathrm{CAR}(\mathcal{H})$ into a graded tensor product [CDF21; AM03] with the factors corresponding to $H = 0$ and $\ker H = \{0\}$, respectively. Moreover, we have restricted our analysis in Theorem 5.2.3 to the cases where either G is abelian or the eigenvalues of H_{SD} are non-degenerate. We believe this can be done in more generality with the interplay between the commutation relations of the unitary operators and the underlying combinatorical fermionic structure.

Chapter 6

Examples from Mathematical Physics

Crossed products appear in several places in mathematical quantum physics, for example in the context of chemical potential in gauge theories [Ara+77], or in the context of the lattice Ising model [AE83]. In the latter situation, one considers the CAR algebra for the lattice $\mathbb{Z}$ (generated by $a^{\#}(\varphi_n)$, $n \in \mathbb{Z}$) and the grading given by $V_{-1}\varphi_n = \varepsilon(n)\varphi_n$ with $\varepsilon(n) = 1$ for $n \geq 1$ and $\varepsilon(n) = -1$ otherwise. This is an example of a non-canonical $\mathbb{Z}_2$-grading γ in the sense that $\gamma_{-1} = \mathrm{Ad}_{V_{-1}} \neq -1$.

In lattice or continuum models with positive Hamiltonian H and positive inverse temperature $\beta > 0$, the Gibbs type condition 3) coincides with the usual Gibbs condition from statistical mechanics. By adding a potential term to the Hamiltonian with suitable growth rate at infinity, one then has many examples in which 3) is satisfied, sometimes only for small enough β (existence of maximal temperature). A detailed investigation of the KMS states of such models and their interpretation is still open.

Here we restrict ourselves to consider another example, a relativistic quantum field theory on two-dimensional Minkowski space-time which can be described by $\mathbb{Z}_2$-crossed products in a non-obvious manner. The model we are considering is often referred to as the Ising QFT. It arises from a scaling limit of the two-dimensional Ising lattice model [MTW77; SMJ78] and turns out to be a completely integrable QFT with two particle S-matrix equal -1 and fits into a larger family of fermionic models [BC21].

To see its relation with $\mathbb{Z}_2$-crossed products, we will describe it from an operator algebraic perspective in which the connection with the Ising lattice model is no longer relevant. The Ising QFT belongs to a family of integrable QFTs that can be analyzed in the framework of algebraic quantum field theory in its vacuum representation (zero temperature). We refer to the review for detailed information about this construction [Lec15], and briefly explain the main points here.

The Hilbert space of the model is the Fermi Fock space $\mathcal{F}_-(\mathcal{H})$ over a one particle space carrying the irreducible unitary positive energy representation U of the proper orthochronous Poincaré group $\mathcal{P}_0(2)$ in two dimensions with mass $m > 0$ and spin zero, i.e. $\mathcal{H} = L^2(\mathbb{R}, d\theta)$. With $p(\theta) := m(\cosh\theta, \sinh\theta)$, the group elements (x, λ) (with $x \in \mathbb{R}^2$ denoting space-time translations and λ the rapidity parameter of a Lorentz boost) are represented as

$$(U(x, \lambda)\psi)(\theta) = e^{ip(\theta)\cdot x}\psi(\theta - \lambda).$$

In particular, the Hamiltonian of this representation is

$$(H\psi)(\theta) = m\cosh(\theta) \cdot \psi(\theta). \tag{6.0.1}$$

As before, we will write α to denote the dynamics given by adjoint action of the second quantization of $U((t, 0), 0) = e^{itH}$ on $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$.

The construction of the QFT proceeds with the help of the fermionic fields Φ and a localization structure. To describe it, we consider the map

$$\mathscr{S}(\mathbb{R}^2) \ni f \longmapsto \hat{f} \in \mathcal{H}, \qquad \hat{f}(\theta) := \tilde{f}(p(\theta)),$$

where $\tilde{f}$ is the Fourier transform of f. The field Φ is then defined using the CAR-algebra by

$$\Phi(\xi) := a^*(\xi) + a(\xi), \quad \xi \in \mathcal{H}.$$

These are not to be confused with the self-dual fields of the preceding chapter as they are only $\mathbb{R}$-linear. For any open subset $\mathcal{O} \subset \mathbb{R}^2$, one then forms the C^*-algebras

$$\mathcal{C}(\mathcal{O}) := C^*(\{\Phi(\hat{f}) \,:\, f \in C^\infty_{c,\mathbb{R}}(\mathcal{O})\}) \subset \mathcal{B}(\mathcal{F}_-(\mathcal{H})).$$

This construction provides us with a net of C^*-algebras transforming covariantly under the second quantization of U, namely $\mathcal{C}(\mathcal{O}_1) \subset \mathcal{C}(\mathcal{O}_2)$ for $\mathcal{O}_1 \subset \mathcal{O}_2$, and $\Gamma(U(g))\mathcal{C}(\mathcal{O})\Gamma(U(g^{-1})) = \mathcal{C}(g\mathcal{O})$ for all $g \in \mathcal{P}_0(2)$. Moreover, the Fock vacuum Ω is cyclic for $\mathcal{C}(\mathcal{O})$ for all open non-empty $\mathcal{O}$.

The net $\mathcal{O} \mapsto \mathcal{C}(\mathcal{O})$ is however not local, i.e. $\mathcal{C}(\mathcal{O}_1)$ and $\mathcal{C}(\mathcal{O}_2)$ do not commute for $\mathcal{O}_1$ spacelike to $\mathcal{O}_2$. This can be understood as a consequence of the Spin-Statistics Theorem, since the field Φ is build from the CAR but U has spin zero. The local content of the Ising QFT is uncovered by realizing a hidden locality property. Namely, for the particular Rindler wedge region $\mathcal{W} := \{(x_0, x_1) \in \mathbb{R}^2 \,:\, x_1 > |x_0|\}$, one finds that the vacuum Ω is also separating for $\mathcal{C}(\mathcal{W})$. Thus Tomita-Takesaki theory applies to the pair $(\mathcal{C}(\mathcal{W})'', \Omega)$, and the modular conjugation $J_{\mathcal{W}}$ turns out to be given by [BL04]

$$J_{\mathcal{W}} = \Gamma(J)\,(-1)^{N(N-1)/2},$$

where $J\xi = \overline{\xi}$ is pointwise complex conjugation on $\mathcal{H} = L^2(\mathbb{R})$. Therefore the second field operator Φ', defined as

$$\Phi'(\xi) := J_{\mathcal{W}}\Phi(J\xi)J_{\mathcal{W}} = (a^*(\xi) - a(\xi))\,(-1)^N,$$

generates the commutant $\mathcal{C}(\mathcal{W})' = C^*(\{\Phi'(\hat{f}) \,:\, f \in C^\infty_{c,\mathbb{R}}(\mathcal{W}')\})''$. Its commutation relations with Φ are given by

$$[\Phi(\xi), \Phi'(\eta)] = 2i \,\mathrm{Im}\langle \xi, \eta\rangle \,(-1)^N, \tag{6.0.2}$$

which vanishes for $\xi = \hat{f}$, $\eta = \hat{g}$ with the supports of f and g spacelike separated. These observations lead to a local QFT by assigning to a double cone $\mathcal{O}_{x,y} := (-\mathcal{W}+x)' \cap (\mathcal{W}+y)'$ (with dashes denoting causal complements) the von Neumann algebra

$$\mathcal{A}(\mathcal{O}_{xy}) := \mathcal{C}'(-\mathcal{W}+x)' \cap \mathcal{C}(\mathcal{W}+y)'.$$

This defines a quantum field theory satisfying all axioms of quantum field theory in its vacuum representation, i.e. at temperature zero. In particular, the vacuum is cyclic and separating for each $\mathcal{A}(\mathcal{O}_{xy})$, the proof of which relies on the split property [Lec05]. It is interacting with S-matrix $(-1)^{N(N-1)/2}$ and asymptotically complete [Lec08].

To connect with crossed products, we consider the two global field algebras $\mathcal{C}(\mathbb{R}^2)$ and $\mathcal{C}'(\mathbb{R}^2)$ generated by the fields Φ and Φ', respectively, and the extended global field algebra

$$\widehat{\mathcal{C}} := C^*(\mathcal{C}(\mathbb{R}^2), \mathcal{C}'(\mathbb{R}^2)),$$

which contains both fields Φ and Φ'.

Theorem 6.0.1. *The field algebras of the two fields Φ, Φ' defining the Ising QFT are $\mathcal{C}(\mathbb{R}^2) = \mathcal{C}'(\mathbb{R}^2) = \mathrm{CAR}(\mathcal{H})$, and the extended field algebra is*

$$\widehat{\mathcal{C}} = C^*(\mathrm{CAR}(\mathcal{H}), (-1)^N) \cong \mathrm{CAR}(\mathcal{H}) \rtimes \mathbb{Z}_2,$$

where the crossed product is taken w.r.t. the canonical grading $\mathrm{Ad}_{(-1)^N}$.

At each inverse temperature $\beta \neq 0$, the C^*-dynamical system $(\widehat{\mathcal{C}}, \alpha)$ consisting of the extended field algebra $\widehat{\mathcal{C}}$ and the dynamics given by the one particle Hamiltonian H (6.0.1) *has a unique KMS state.*

Proof. The inclusion $\mathcal{C}(\mathbb{R}^2) \subset \mathrm{CAR}(\mathcal{H})$ holds by definition. Passing to complex linear test functions and limits, one notes $a^*(\hat{f}) + a(\hat{\overline{f}}) \in \mathcal{C}(\mathbb{R}^2)$ for any (complex) $f \in \mathscr{S}(\mathbb{R}^2)$. Hence one can choose f in such a way that $\hat{\overline{f}} = 0$, and obtain a dense set of vectors $\hat{f} \in \mathcal{H}$ in this way. This shows the opposite inclusion $\mathrm{CAR}(\mathcal{H}) \subset \mathcal{C}(\mathbb{R}^2)$.

In view of the definition of Φ', we clearly have $\mathcal{C}'(\mathbb{R}^2) \cong \mathcal{C}(\mathbb{R}^2)$.

The extended global field algebra $\widehat{\mathcal{C}}$ contains $\mathrm{CAR}(\mathcal{H})$ and, in view of (6.0.2), also the grading operator $(-1)^N$. Hence $\widehat{\mathcal{C}} = C^*(\mathrm{CAR}(\mathcal{H}), (-1)^N)$.

As $\dim \mathcal{H} = \infty$, this grading operator is not contained in $\mathrm{CAR}(\mathcal{H})$ [Ara71]. Taking into account that $\mathrm{CAR}(\mathcal{H})$ is simple, we may apply Lemma 4.1.5 and conclude $C^*(\mathrm{CAR}(\mathcal{H}), (-1)^N) \cong \mathrm{CAR}(\mathcal{H}) \rtimes \mathbb{Z}_2$.

The statement about existence and uniqueness of KMS states follows by application of Theorem 5.2.3: The Hamiltonian H (6.0.1) has continuous spectrum and therefore violates the Gibbs type condition 3). Hence the unique (α, β)-KMS state of $\mathrm{CAR}(\mathcal{H})$ has a unique extension to $\widehat{\mathcal{C}}$. □

We note that all n-point functions of the KMS state $\hat{\omega}_\beta$ of $\widehat{\mathcal{C}}$ (in both field types) can be read off from our construction in Section 5. It should furthermore be noted that $\widehat{\mathcal{C}}$ contains many even local observables, because the even part of $\mathcal{A}(\mathcal{O}_{xy})$ is generated by even polynomials in the field Φ [BS07]. Nonetheless, the field algebra $\widehat{\mathcal{C}}$ differs from the quasilocal C^*-algebra $\mathfrak{A}$ of the Ising QFT in its odd elements, so that the KMS state constructed here describes only parts of the Ising QFT at finite temperature $\beta^{-1} > 0$. In conclusion, we note that also the quasilocal C^*-algebra $\mathfrak{A}$ can be shown to have KMS states by combining the results of [BJ89; Lec05]. We leave a more detailed investigation of their relation to a future work.

Chapter 7

Summary and Outlook

In this thesis, we have examined the KMS states on the crossed product of a C^*-algebra $\mathcal{A}$ by a finite group G. We have found that every twist invariant KMS state ω on $\mathcal{A}$ allows for a canonical extension $\hat{\omega}^{\text{can}}$. The non-uniqueness of $\hat{\omega}^{\text{can}}$ can be characterized by a variety of related objects: The center of the von Neumann crossed product, families of twisted KMS functionals and operator-valued states (Thm. 4.5.6).

These concepts are studied in detail for fermionic systems. In the application to the (self-dual) CAR algebra, the underlying connection between the spectral properties of the Hamiltonian and non-vanishing twisted KMS functionals is made precise. In this case the Gibbs type condition

$$\mathrm{Tr}_{\mathcal{K}_\perp}(e^{-|\beta H_\perp|}) < \infty$$

holds if and only if there exists a non-vanishing twisted KMS functional of $\mathrm{CAR}_{\mathrm{SD}}(\mathcal{K}, \Gamma)$ dominated by ω. We then show that these functionals form a positively compatible and covariant family of twisted KMS functionals in case G is abelian. This might hold as well for more general groups and should be analyzed in a future work. Our derivation of the twisted KMS functionals was done for unitaries of finite order. This assumption can certainly be weakened and thus generalizations to other groups, especially $U(1)$, are feasible. These results were then applied to the thermal state of the extended field algebra of the Ising QFT. Although we have calculated the KMS states of the crossed product of a fermionic system, we have not yet given a physical interpretation of these states. We believe that, in case non-vanishing twisted functionals exist, the corresponding KMS states on the crossed product show a kind of generalized statistics. In particular, the relation between the canonical $\mathbb{Z}_2$-twist $\gamma = -1$ and bosonization is worth investigating.

The focus of this thesis was on crossed products by finite and thus amenable groups. A generalization of this work to crossed products by discrete groups would thus require the discussion of amenability, see [BK18; Urs21]. Another possible direction would be to specialize to abelian groups G. We have started considerations in this direction in Corollary 4.6.7. This viewpoint has also been considered in [CT21] for traces on $\mathcal{A}$ and KMS states on $\mathcal{A} \rtimes_\gamma G$. This abelianess of G furthermore allows for the application of Fourier theory to our question at hand. On a similar note, we have seen that the existence of an $\mathfrak{M}$-valued unitary representation allows for the decomposition of $\mathfrak{M}$-valued covariant inner states. It is however an open question under which conditions the individual twisted centers allow for a construction of an $\mathfrak{M}$-valued (partial) unitary representation. A consideration worth investigating in this direction is the relation to projective unitary representations.

In the introductory section to KMS states, we recall the convex decomposition theory of KMS states and especially the correspondence between extremality in the set of KMS states and factoriality of the associated von Neumann algebra. Due to this correspondence, we have mostly studied extensions of extremal KMS states to the crossed product. It remains to be shown that the decomposition theory of KMS states on $\mathcal{A}$ and $\mathcal{A} \rtimes_\gamma G$ respect these decompositions.

Regarding physical applications, it was briefly mentioned in the introduction as well as Section 6 that the $\mathbb{Z}_2$-crossed product of the CAR-algebra can be used to study the lattice Ising model. A generalization of said model is the Potts model [Wu82], where the spin Hilbert space has arbitrary but finite dimension. This system is mathematically well-behaved while still showing interesting physical features [CGN01; Che22]. Hence, studying the thermal behavior of the Potts model from the viewpoint of crossed products would be of interest.

Appendix A

Appendix

Complex Analysis

Theorem A.1 (Liouville's Theorem)**.** *[FB06, Theorem 3.7]*

Every bounded entire analytic function is constant.

Theorem A.2 (Schwarz Reflection Principle)**.** *[Lan75, Theorem 13.1.1]*

Let U^+ be an open set in $\mathbb{C}^+$ and $\mathbb{R}$ contained in the closure of U^+. Denote by U^- the reflection along the real axis and $U = U^+ \cup \mathbb{R} \cup U^-$.

If f is a function on $U^+ \cup \mathbb{R}$, analytic on U^+ and real-valued, continuous on $\mathbb{R}$, then f has a unique analytic continuation F to U, and

$$F(z) = \overline{f(\overline{z})}, \quad z \in U^-.$$

Theorem A.3. *[FB06, Theorem 3.2]*

Let $\mathcal{O} \subset \mathbb{C}$ be an open and connected set and suppose $f, g : \mathcal{O} \to \mathbb{C}$ are analytic functions which coincide on a subset $S \subset \mathcal{O}$ with an accumulation point in $\mathcal{O}$. Then $f = g$.

Operator Algebras and Dynamical Systems

Proposition A.4. *[Tak01, Proposition 3.12]*

Let $\mathcal{A}$ be a W^*-algebra and ω a normal positive linear functional on $\mathcal{A}$. Then the corresponding GNS representation $(\pi, \mathcal{H})$ is normal and in particular $\pi(\mathcal{A})$ is a von Neumann algebra and π is σ-weakly continuous.

Theorem A.5. *[BR87, Section 2.5.3]*

Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system. Then the set of entire analytic elements $\mathcal{A}_\alpha$ is norm-dense in $\mathcal{A}$.

Let $(\mathfrak{M}, \alpha)$ be a W^*-dynamical system. Then the set of entire analytic elements $\mathfrak{M}_\alpha$ is a weakly-dense and strongly-dense subspace of $\mathfrak{M}$.

Definition A.6. *[BR97, Definition 5.3.1]*

Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system and $\gamma \in \operatorname{Aut} \mathcal{A}$ (not necessarily commuting with α). A state $\omega \in \mathcal{S}(\mathcal{A})$ is called α-KMS state at inverse temperature $\beta \in \mathbb{R}$, or (α, β)-KMS state, if

$$\omega(a\alpha_{i\beta}(b)) = \omega(ba), \qquad a, b \in \mathcal{B},$$

where $\mathcal{B} \subset \mathcal{A}_\alpha$ is a norm-dense *-subalgebra. A functional ρ on $\mathcal{A}$ is called γ-twisted α-KMS functional at inverse temperature $\beta \in \mathbb{R}$, if it is continuous and

$$\rho(a\alpha_{i\beta}(b)) = \rho(b\gamma(a)), \qquad a, b \in \mathcal{B},$$

where $\mathcal{B} \subset \mathcal{A}_\alpha$ is a norm-dense *-subalgebra.

Let $(\mathfrak{M}, \alpha)$ be a W^*-dynamical system. A state $\omega \in \mathcal{S}(\mathfrak{M})$ is called α-KMS state at inverse temperature $\beta \in \mathbb{R}$, or (α, β)-KMS state, if ω is normal and

$$\omega(x\alpha_{i\beta}(y)) = \omega(yx), \qquad x, y \in \mathcal{N},$$

where $\mathcal{N} \subset \mathfrak{M}_\alpha$ is a strongly dense *-subalgebra.

Proofs

Proof. This passage completes the proof of Proposition 5.2.1

Given the recursion relation

$$\rho_\beta(\Phi_{\xi_1}\cdots\Phi_{\xi_{2n+2}}) = \sum_{j=2}^{2n+2}(-1)^j\langle\Gamma\xi_1, C_\beta^V\xi_j\rangle\cdot\rho_\beta(\Phi_{\xi_2}\cdots\widehat{\Phi}_{\xi_j}\cdots\Phi_{\xi_{2n+2}}),$$

we use the induction hypothesis to expand the $2n$-point function in terms of scalar products in order to show that ρ_β is indeed quasifree. To write down the combinatorics more easily, we make the following distinction:

$$\rho_\beta(\Phi(\xi_2)\cdots\widehat{\Phi}(\xi_j)\cdots\Phi(\xi_{2n+2})) = \rho_\beta(\mathbb{1})(-1)^{\frac{(n-1)n}{2}}\times$$
$$\begin{cases}\sum_{\sigma_j}\operatorname{sgn}(\sigma_j)\prod_{i=2}^{j-1}\langle\Gamma\xi_{\sigma_j(i)}, C_\beta^V\xi_{\sigma_j(i+n+1)}\rangle\prod_{i=j+1}^{n+2}\langle\Gamma\xi_{\sigma_j(i)}, C_\beta^V\xi_{\sigma_j(i+n)}\rangle & \text{if } j\le n+1\\ \sum_{\sigma_j}\operatorname{sgn}(\sigma_j)\prod_{i=2}^{j-n-1}\langle\Gamma\xi_{\sigma_j(i)}, C_\beta^V\xi_{\sigma_j(i+n)}\rangle\prod_{i=j-n}^{n+1}\langle\Gamma\xi_{\sigma_j(i)}, C_\beta^V\xi_{\sigma_j(i+n+1)}\rangle & \text{if } j> n+1\end{cases}$$

where the sum runs over all permutations σ_j of $\{2,\cdots,j-1,j+1,\cdots,2n+2\}$ satisfying the following j-dependent conditions.
For $j\le n+1$:

$$\sigma_j(2)<\sigma_j(3)<\cdots\sigma_j(j-1)<\sigma_j(j+1)<\cdots<\sigma_j(n+2)$$
$$\sigma_j(i)<\sigma_j(i+n+1)\quad\forall\, 2\le i<j$$
$$\sigma_j(i)<\sigma_j(i+n)\quad\forall\, j<i\le n+2$$

For $j>n+1$:

$$\sigma_j(2)<\sigma_j(3)<\cdots<\sigma_j(n+1)$$
$$\sigma_j(i)<\sigma_j(i+n)\quad\forall\, 2\le i\le j-(n+1)$$
$$\sigma_j(i)<\sigma_j(i+n+1)\quad\forall\, j-(n+1)<i\le n+1$$

Note that $\operatorname{sgn}(\sigma_j)$ is the sign of σ_j as a permutation of $2n$ elements.

For such a permutation σ_j of $2n$ elements, we will now construct a unique permutation τ_j of $2n+2$ elements which still pairs the same elements as σ_j and further pairs 1 and j.
Firstly, consider the case of $j\le n+1$. Define the permutation τ_j by

$$\begin{pmatrix}1 & 2 & \dots & j-1 & j & \dots & n+1 & n+2 & \dots & 2n+2\\ 1 & \sigma_j(2) & \dots & \sigma_j(j-1) & \sigma_j(j+1) & \dots & \sigma_j(n+2) & j & \dots & \sigma_j(2n+2)\end{pmatrix}.$$

It satisfies $\tau_j(i)<\tau_j(i+1)$ for all $i<n+1$ and $\tau_j(i)<\tau(i+n+1)$ for all $i\le n+1$. Moreover, it pairs 1 and j, since $\tau_j(1)=1$ and $\tau_j(n+2)=j$, and respects the pairings of σ_j.

Next we show that τ_j is the unique extension of σ_j pairing 1 and j and satisfying the properties

$$\tau_j(1) < \tau_j(2) < \cdots < \tau_j(n+1) \quad \text{and} \quad \tau_j(i) < \tau_j(i+n+1) \quad \forall\, 1 \leq i \leq n+1.$$

Let τ be another extension of σ_j that pairs 1 and j and satisfies these properties. Then $\tau(1)$ is smaller then $\tau(i)$ for all $2 \leq i \leq n+1$. These in turn satisfy $\tau(i) < \tau(i+n+1)$ and therefore $\tau(1)$ is smaller than all other $2n+1$ elements, which implies $\tau(1) = 1$. Since τ pairs 1 and j, this already implies $\tau(n+2) = j$. Similarly, since τ is an extension of σ_j and satisfies $\tau(1) < \cdots < \tau(n+1)$, the remaining part of the permutation is fixed. Thus it is given by τ_j.
The signs of σ_j and τ_j are related by

$$\operatorname{sgn}(\sigma_j) = (-1)^{n-j} \operatorname{sgn}(\tau_j).$$

This can be seen by counting the number of neighboring transpositions needed to permute j to the j-th position.

Secondly, consider the case of $j > n+1$. We similarly define τ_j by

$$\begin{pmatrix} 1 & 2 & \ldots & n+2 & n+3 & \ldots & j & j+1 & \ldots & 2n+2 \\ 1 & \sigma_j(2) & \ldots & j & \sigma_j(n+2) & \ldots & \sigma_j(j-1) & \sigma_j(j+1) & \ldots & \sigma_j(2n+2) \end{pmatrix}.$$

It satisfies $\tau_j(i) < \tau_j(i+1)$ for all $i < n+1$ and $\tau_j(i) < \tau(i+n+1)$ for all $i \leq n+1$. Moreover, it pairs 1 and j, since $\tau_j(1) = 1$ and $\tau_j(n+2) = j$, and respects the pairings of σ_j.
Next we show again that τ_j is the unique extension of σ_j pairing 1 and j and satisfying the properties

$$\tau_j(1) < \tau_j(2) < \cdots < \tau_j(n+1) \quad \text{and} \quad \tau_j(i) < \tau_j(i+n+1) \quad \forall\, 1 \leq i \leq n+1.$$

Let τ be another extension of σ_j that pairs 1 and j and satisfies these properties. Then $\tau(1)$ is smaller then $\tau(i)$ for all $2 \leq i \leq n+1$. These in turn satisfy $\tau(i) < \tau(i+n+1)$ and therefore $\tau(1)$ is smaller than all other $2n+1$ elements, which implies $\tau(1) = 1$. Since τ pairs 1 and j, this already implies $\tau(n+2) = j$. Similarly, since τ is an extension of σ_j and satisfies $\tau(1) < \cdots < \tau(n+1)$, the remaining part of the permutation is fixed. Thus it coincides with τ_j.
The signs of σ_j and τ_j are again related by

$$\operatorname{sgn}(\sigma_j) = (-1)^{n-j} \operatorname{sgn}(\tau_j),$$

which can be seen by the same argument as above.

Coming back to the expansion of the $(2n+2)$-point function, we have for $j \leq n+1$ that

$$(-1)^j \langle \Gamma\xi_1, C_\beta^V \xi_j \rangle \operatorname{sgn}(\sigma_j) \prod_{i=2}^{j-1} \langle \Gamma\xi_{\sigma_j(i)}, C_\beta^V \xi_{\sigma_j(i+n+1)} \rangle \prod_{i=j+1}^{n+2} \langle \Gamma\xi_{\sigma_j(i)}, C_\beta^V \xi_{\sigma_j(i+n)} \rangle$$
$$= (-1)^n \operatorname{sgn}(\tau_j) \prod_{i=1}^{n+1} \langle \Gamma\xi_{\tau_j(i)}, C_\beta^V \xi_{\tau_j(i+n+1)} \rangle$$

Similarly, for $j > n+1$ we have

$$(-1)^j \langle \Gamma\xi_1, C_\beta^V \xi_j \rangle \operatorname{sgn}(\sigma_j) \prod_{i=2}^{j-n-1} \langle \Gamma\xi_{\sigma_j(i)}, C_\beta^V \xi_{\sigma_j(i+n)} \rangle \prod_{i=j-n}^{n+1} \langle \Gamma\xi_{\sigma_j(i)}, C_\beta^V \xi_{\sigma_j(i+n+1)} \rangle$$
$$= (-1)^n \operatorname{sgn}(\tau_j) \prod_{i=1}^{n+1} \langle \Gamma\xi_{\tau_j(i)}, C_\beta^V \xi_{\tau_j(i+n+1)} \rangle$$

Combining these results, we have

$$\begin{aligned}
&\rho_\beta(\Phi(\xi_1)\cdots\Phi(\xi_{2n+2})) \\
&= \rho_\beta(\mathbb{1})(-1)^{\frac{(n-1)n}{2}} \left(\sum_{j=2}^{n+1} \sum_{\tau_j} (-1)^n \operatorname{sgn}(\tau_j) \prod_{i=1}^{n+1} \langle \Gamma\xi_{\tau_j(i)}, C_\beta^V \xi_{\tau_j(i+n+1)} \rangle \right. \\
&\quad \left. + \sum_{j=n+2}^{2n+2} \sum_{\tau_j} (-1)^n \operatorname{sgn}(\tau_j) \prod_{i=1}^{n+1} \langle \Gamma\xi_{\tau_j(i)}, C_\beta^V \xi_{\tau_j(i+n+1)} \rangle \right) \\
&= \rho_\beta(\mathbb{1})(-1)^{\frac{(n-1)n}{2}} \sum_{\tau} (-1)^n \operatorname{sgn}(\tau) \prod_{i=1}^{n+1} \langle \Gamma\xi_{\tau(i)}, C_\beta^V \xi_{\tau(i+n+1)} \rangle \\
&= \rho_\beta(\mathbb{1})(-1)^{\frac{(n+1)n}{2}} \sum_{\tau} \operatorname{sgn}(\tau) \prod_{i=1}^{n+1} \langle \Gamma\xi_{\tau(i)}, C_\beta^V \xi_{\tau(i+n+1)} \rangle.
\end{aligned}$$

This completes the induction and shows that ρ_β is, up to the prefactor $\rho_\beta(\mathbb{1})$, a quasifree functional. □

Bibliography

[AB83] H. Araki and E. Barouch. "On the dynamics and ergodic properties of the XY model". *Journal of Statistical Physics* 31 (1983), 327–345.

[AE83] H. Araki and D. E. Evans. "On a C*-algebra approach to phase transition in the two-dimensional Ising model". *Comm. Math. Phys.* 91 (1983), 489–503.

[AM03] H. Araki and H. Moriya. "Joint extension of states of subsystems for a CAR system". *Commun. Math. Phys.* 237 (2003), 105–122.

[Ara+77] H. Araki, D. Kastler, M. Takesaki, and R. Haag. "Extension of KMS states and chemical potential". *Comm. Math. Phys.* 53.2 (1977), 97–134.

[Ara68] H. Araki. "On the diagonalization of a bilinear Hamiltonian by a Bogoliubov transformation". *Publ. Res. Inst. Math. Sci. Kyoto University, Ser. A* 4 (1968), 387–412.

[Ara71] H. Araki. "On Quasifree States of CAR and Bogoliubov Automorphisms". *Publ. Res. Inst. Math. Sci. Kyoto* 6 (1971), 385–442. DOI: 10.2977/prims.

[Ara84] H. Araki. "On the XY-model on two-sided infinite chain". *Publications of the Research Institute for Mathematical Sciences* 20.2 (1984), 277–296.

[AW64] H. Araki and W. Wyss. "Representations of canonical anticommutation relations". *Helv. Phys. Acta* 37 (1964), 136.

[BC21] H. Bostelmann and D. Cadamuro. "Fermionic integrable models and graded Borchers triples". *arXiv preprint* (2021). DOI: 10.48550/arXiv.2112.14686.

[BG07] D. Buchholz and H. Grundling. "Algebraic Supersymmetry: A case study". *Comm. Math. Phys.* 272 (2007), 699–750. DOI: 10.1007/s00220-006-0177-z.

[BJ89] D. Buchholz and P. Junglas. "On the Existence of Equilibrium States in Local Quantum Field Theory". *Comm. Math. Phys.* 121 (1989), 255–270.

[BJL02] H. Baumgärtel, M. Jurke, and F. Lledó. "Twisted duality of the CAR algebra". *J. Math. Phys.* 43 (2002), 4158–4179. DOI: 10.1063/1.1483376.

[BK18] R. S. Bryder and M. Kennedy. "Reduced twisted crossed products over C*-simple groups". *International Mathematics Research Notices* 2018.6 (2018), 1638–1655.

[BL04] D. Buchholz and G. Lechner. "Modular nuclearity and localization". *Ann. Henri Poincaré* 5 (2004), 1065–1080. DOI: 10.1007/s00023-004-0190-8.

[BL99] D. Buchholz and R. Longo. “Graded KMS functionals and the breakdown of supersymmetry”. *Adv. Theor. Math. Phys.* 3 (1999), 615–626.

[BMV68] E. Balslev, J. Manuceau, and A. Verbeure. “Representations of anticommutation relations and Bogolioubov transformations”. *Comm.Math. Phys.* 8 (1968), 315–326.

[BR87] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics I.* Springer, 1987.

[BR97] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics II.* Springer, 1997.

[BS07] D. Buchholz and S. J. Summers. “String- and brane-localized fields in a strongly nonlocal model”. *J. Phys.* A40 (2007), 2147–2163. DOI: 10.1088/1751-8113.

[CDF21] V. Crismale, R. Duvenhage, and F. Fidaleo. “C^*-Fermi systems and detailed balance”. *Anal. Math. Phys.* 11 (2021). 44 pages. DOI: 10.1007/s13324-020-00412-0.

[CGN01] M. Caselle, F. Gliozzi, and S. Necco. “Thermal operators and cluster topology in the q-state Potts model”. *Journal of Physics A: Mathematical and General* 34.3 (Jan. 2001), 351. DOI: 10.1088/0305-4470/34/3/302.

[Cha+23] V. Chandrasekaran, R. Longo, G. Penington, and E. Witten. “An algebra of observables for de Sitter space”. *Journal of High Energy Physics* 2023.2 (2023), 1–56.

[Che22] N. Chepiga. “Critical properties of quantum three-and four-state Potts models with boundaries polarized along the transverse field”. *SciPost Physics Core* 5.2 (2022), 031.

[CT21] J. Christensen and K. Thomsen. “KMS states on crossed products by abelian groups”. *Math. Scand.* 127.3 (2021). DOI: 10.48550/arXiv.2006.14443.

[DG22] J. Dereziski and C. Gérard. *Mathematics of Quantization and Quantum Fields.* Cambridge University Press, 2022.

[DKR66] S. Doplicher, D. Kastler, and D. W. Robinson. “Covariance Algebras in Field Theory and Statistical Mechanics ”. *Comm. Math. Phys.* 3 (1966), 1–28.

[FB06] E. Freitag and R. Busam. *Funktionentheorie I.* 4th ed. Springer, 2006.

[Few+24] C. J. Fewster, D. W. Janssen, L. D. Loveridge, K. Rejzner, and J. Waldron. “Quantum reference frames, measurement schemes and the type of local algebras in quantum field theory”. *arXiv preprint arXiv:2403.11973* (2024).

[HHW67] R. Haag, N. M. Hugenholtz, and M. Winnink. “On the Equilibrium States in Quantum Statistical Mechanics ”. *Comm. Math. Phys.* 5 (1967), 215–236.

[Hil15] R. Hillier. “Super-KMS functionals for graded-local conformal nets”. *Ann. Henri Poincaré* 16 (2015), 1899–1936.

[JL93] S. Y. Jang and S. G. Lee. “Simplicity of crossed products of *-algebras”. *Proceedings of the American Mathematical Society* 118.3 (1993), 823–826.

[JLW89] A. Jaffe, A. Lesniewski, and M. Wisniowski. "Deformations of super-KMS functionals". *Comm. Math. Phys.* 121 (1989), 527–540.

[Kal69] R. R. Kallman. "A generalization of free action". *Duke Mathematical Journal* 36.4 (1969), 781–789. DOI: 10.1215/S0012-7094-69-03693-X.

[Kas89] D. Kastler. "Cyclic cocycles from graded KMS functionals". *Comm. Math. Phys.* 121 (1989), 345–350.

[KL24] M. S. Klinger and R. G. Leigh. "Crossed products, extended phase spaces and the resolution of entanglement singularities". *Nuclear Physics B* 999 (2024), 116453. DOI: https://doi.org/10.1016/j.nuclphysb.2024.116453.

[Kub57] R. Kubo. "Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems". *Journal of the physical society of Japan* 12.6 (1957), 570–586.

[Lan75] S. Lang. $SL_2(R)$. Springer, 1975.

[Lec05] G. Lechner. "On the existence of local observables in theories with a factorizing S-matrix". *J. Phys.* A38 (2005), 3045–3056. DOI: 10.1088/0305-4470.

[Lec08] G. Lechner. "Construction of Quantum Field Theories with Factorizing S-Matrices". *Comm. Math. Phys.* 277 (2008), 821–860. DOI: 10.1007/s00220-007-0381-5.

[Lec15] G. Lechner. "Algebraic Constructive Quantum Field Theory: Integrable Models and Deformation Techniques". *Advances in Algebraic Quantum Field Theory*. Ed. by R. Brunetti et al. Springer, 2015, 397–449.

[LN03] M. Laca and S. Neshveyev. *KMS states of quasi-free dynamics on Pimsner algebras*. Apr. 2003.

[LZ24] K. Li and J. Zhang. *Tracial states on groupoid C^*-algebras and essential freeness*. Jan. 2024. DOI: 10.48550/arXiv.2401.15546.

[MS59] P. C. Martin and J. Schwinger. "Theory of many-particle systems. I". *Physical Review* 115.6 (1959), 1342.

[MTW77] B. M. McCoy, C. A. Tracy, and T. T. Wu. "Two-Dimensional Ising Model as an Exactly Solvable Relativistic Quantum Field Theory: Explicit Formulas for n Point Functions". *Phys. Rev. Lett.* 38 (1977), 793–796. DOI: 10.1103/PhysRevLett.38.793.

[Mur90] G. J. Murphy. *C*-Algebras and Operator Theory*. Academic Press, 1990.

[Nee11] K.-H. Neeb. *Holomorphy and convexity in Lie theory*. Vol. 28. Walter de Gruyter, 2011.

[Nes14] S. Neshveyev. "KMS states on the C*-algebras of non-principal groupoids". *J. Operator Theory* 70.2 (2014). 12 pages; subsumes the earlier preprint "Traces on crossed products" arXiv:1010.0600; minor correction of the version published in JOT, 513–530. DOI: 10.48550/arXiv.1106.5912.

[NS22] S. Neshveyev and N. Stammeier. "The groupoid approach to equilibrium states on right LCM semigroup C^*-algebras". *J. Lond. Math. Soc.* 105.1 (2022), 220–250.

[OP78] D. Olesen and G. K. Pedersen. "Applications of the Connes spectrum to C-dynamical systems". *Journal of Functional Analysis* 30.2 (1978), 179–197.

[SGL24] R. C. da Silva, J. Grosse, and G. Lechner. "KMS states on $\mathbb{Z}_2$-crossed products and twisted KMS functionals". *arXiv preprint arXiv:2402.15574* (2024).

[SMJ78] M. Sato, T. Miwa, and M. Jimbo. "Holonomic quantum fields I". *Publ. Res. Inst. Math. Sci.* 14.1 (1978), 223–267.

[Tak01] M. Takesaki. *Theory of Operator Algebras I*. Springer, 2001.

[Tak03] M. Takesaki. *Theory of Operator Algebras II*. Springer, 2003.

[Tak67] M. Takesaki. "Covariant representations of C*-algebras and their locally compact automorphism groups". *Acta Mathematica* 119.1 (1967), 273. DOI: 10.1007/BF02392085.

[Tho23] K. E. Thomsen. "An introduction to KMS weights". *arXiv preprint* (2023). DOI: 10.48550/arXiv.2204.01125.

[Urs21] D. Ursu. "Characterizing traces on crossed products of noncommutative C*-algebras". *Adv. Math.* 391 (2021), 107955. DOI: 10.1016/j.aim.2021.107955.

[Wil07] D. P. Williams. *Crossed products of C*-algebras*. Providence, Rhode Island: American Mathematical Society, 2007.

[Wu82] F.-Y. Wu. "The potts model". *Reviews of modern physics* 54.1 (1982), 235.

Acknowledgments

I would like to take this page to thank all those, whose academic and personal support made this work possible.

This thesis was only made possible by the excellent supervision of Prof. Gandalf Lechner. I greatly appreciate the time and effort he spent guiding me throughout this work. I am especially thankful for the numerous opportunities that he provided, to present my research and come in contact with physicists and mathematicians alike.

I would like to extend my sincere thanks to Prof. Hanno Sahlmann, who already supervised my Bachelor's Thesis three years ago. Only due to his early influence am I now working on questions of mathematical physics. I appreciate your open ear, especially our discussion on quantum theory were a pleasure to me.

In addition, my thanks go to Dr. Ricardo Correa da Silva who guided me throughout my work. Our discussions in front of the blackboard are invaluable to this thesis. I would further like to acknowledge Prof. Karl-Hermann Neeb, whose lectures influenced me since the beginning of my mathematics journey as well as Prof. Kang Li for the stimulating discussions on operator algebras. My gratitude extends to the all members of the mathematical physics group at FAU not mentioned so far. The open minded discussions about physics, mathematics and everything beyond were a pleasure to me.

Last but not least, I would like to express my sincere gratitude to my family, my partner Franziska Kluge and my friends. My studies and my academic work would not have been possible without your emotional and academic support.

Declaration

I hereby certify that I have written this thesis independently and that I have not used any sources or aids other than those indicated, that all passages of the work which have been taken over verbatim or in spirit from other sources have been marked as such and that the work has not yet been submitted to any examination authority in the same or a similar form.

Erlangen, August 23, 2024

This thesis from the field of mathematical physics is submitted in a similar form as a Master's thesis in both the Master's degree program in Mathematics and the Master's degree program in Physics.